SUMMER SCHOOL 2025

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Substitution Rule

Recall FTC-1

If f is cont. on $[a,b]$, and $g(x) = \int_a^x f(t)\,dt$ on $[a,b]$

Then g is also cont. on $[a,b]$, diff'able on (a,b)

and $g'(x) = f(x)$

i.e. any cont. function has antideriv.

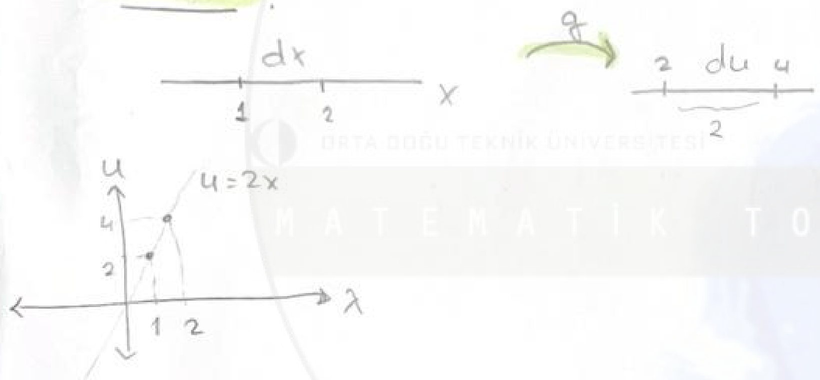
SR for Indefinite Integrals

Suppose $u = g(x)$ is cont., diff'able whose range is contained is an interval I and f is cont. on I

Then

$$\int f(\underbrace{g(x)}_{u}) \cdot \underbrace{g'(x)\,dx}_{du} = \int f(u)\,du$$

remark:



eg: $u = g(x) = 2x \quad du = 2dx$

$\Downarrow$

$g'(x) = 2$

proof Let $F' = f$ that is, let F be an antideriv. of f on I
(existence of F is guaranted by FTC(1))

Then

$$\text{Left-handside} = \int F'(g(x))g'(x)\,dx \overset{\text{by chain rule}}{=} \int \frac{d}{dx}\left(F(g(x)\right)dx \overset{\text{FTC-1}}{=} F(g(x)) + C$$

$$= F(u) + C \overset{\text{FTC}}{=} \int f(u)\,du = \text{right handside}$$

SR for Definite Integrals

Suppose $u = g(x)$ is cont. diff'able function whose range is an interval I, and $f(x)$ is cont. on I. Assume $[a,b] \subset I$

Then

$$\underbrace{\int_a^b f(g(x))\, g'(x)\, dx}_{LHS} = \underbrace{\int_{g(a)}^{g(b)} f(u)\, du}_{RHS}$$

proof Let F be an antiderivative of f on I (So, $F' = f$ on I)

Then

$$RHS \overset{FTC\text{-}2}{=} F(g(b)) - F((g(a)) = (F\circ g)(b) - (F\circ g)(a) = F(g(x)) \Big|_a^b$$

observe FTC -1 $\displaystyle\int_a^b f(g(x))g'(x)\, dx = LHS$

$$\frac{d}{dx}\left(F(g(x))\right) = F'(g(x))\, g'(x) = f(g(x))\, g'(x)$$

examples

compute each

① $I = \displaystyle\int x^2 (x^3+5)^{114}\, dx$

soln: Let $u = x^3+5 \Rightarrow du = 3x^2\, dx \Rightarrow I = \displaystyle\int \frac{1}{3} u^{114}\, du = \frac{1}{3}\frac{u^{115}}{115} + C \overset{\text{back substitution}}{=} \frac{1}{345}(x^3+5)^{115} + C$

② $I = \displaystyle\int \frac{\ln^2 x}{x}\, dx$

Solution: Let $u = \ln x$, $du = \dfrac{dx}{x}$ $\Rightarrow I = \displaystyle\int u^2\, du = \frac{u^3}{3} + C \overset{\text{back subst.}}{\Rightarrow} \frac{\ln^3(x)}{3} + C$

③ $I = \displaystyle\int_0^{13} \frac{1}{\sqrt[3]{(1+2x)^2}}\, dx$

Solution: Let $u = 1+2x$, $du = 2dx$ $\Rightarrow I = \dfrac{1}{2}\displaystyle\int_{x=0}^{x=13} u^{-2/3}\, du$ $\left(\begin{array}{l} x = 13 \\ u = 1+(2.13) \\ x = 0 \\ u = 1+(2.0) \end{array}\right) \Rightarrow I = u^{-1/3}\Big|_1^{27} = 3$

④ $I = \int \frac{1}{a^2+x^2}\,dx$ where $a \neq 0$ fixed constant

Solution

observe $I = \int \frac{dx}{a^2\left(1+\frac{x^2}{a^2}\right)} = \underbrace{\frac{1}{a^2}}_{\text{some constant}} \int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx$ Let $u = \frac{x}{a}$

$du = \frac{1}{a}\,dx$

then $I = \frac{1}{a}\int \frac{\overbrace{\frac{1}{a}\,dx}^{du}}{1+\underbrace{\left(\frac{x}{a}\right)}_{u}^2} = \frac{1}{a}\int \frac{du}{1+u^2} = \frac{1}{a}\tan^{-1}(u)+c \overset{\text{(back subst)}}{=} \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

⑤ $I = \int \frac{dx}{e^x+e^{-x}}$

Soln

observe $I = \int \frac{dx}{e^{-x}(e^{2x}+1)} = \int \frac{e^x}{e^{2x}+1}\,dx$ Let $u = e^x$

$du = e^x dx \Rightarrow I = \int \frac{du}{u^2+1}$

$= \tan^{-1}(u)+c = \tan^{-1}(e^x)+c$

⑥ $I = \int \frac{dx}{(e^x+1)}$

Soln Let $e^x = \frac{1}{t}$ $\Rightarrow I = \int \frac{-\frac{1}{t}\,dt}{\left(\frac{1}{t}+1\right)} = -\int \frac{1}{t+1}\,dt$ say $t+1=u$, $dt = du$ $\Rightarrow -\int \frac{du}{u} = -\ln|u|+c$

$e^x dx = -t^{-2}dt$

$\frac{1}{t} \Rightarrow dx = -t^{-1}dt$

$-\ln|u|+c = -\ln|t+1|+c = -\ln|1+e^{-x}|+c$

⑦ $I = \int \underbrace{\sin(\cos(x))}_{u} \cdot \underbrace{\cos(\cos(x))\sin x\,dx}_{-du}$

Soln

Let $u = \sin(\cos(x))$

$du = \cos(\cos(x)).(-\sin(x))\,dx$

$\Rightarrow I = -\int u\,du = \frac{-u^2}{2}+c = \frac{-\sin^2(\cos(x))}{2}+c$

8) $I = \int \tan x \, dx$

Soln:

observe $I = \int \frac{\sin x}{\cos x} dx$ Let $u = \cos x$, $du = -\sin x \, dx$ $\Rightarrow I = \int \frac{-du}{u} = -\ln|u| + c$

$\begin{pmatrix} \text{back} \\ \text{subst.} \end{pmatrix} = -\ln|\cos x| + c$

9) $I = \int \sec\omega \, d\omega$

solution

$$I = \int \frac{\sec\omega(\sec\omega + \tan\omega)}{(\sec\omega + \tan\omega)} d\omega = \int \frac{\overbrace{(\sec^2\omega + \sec\omega\tan\omega)\, d\omega}^{du}}{\underbrace{\sec\omega + \tan\omega}_{u}}$$

Let $\sec\omega + \tan\omega = u$

$(\sec\omega\tan\omega + \sec^2\omega)\, d\omega = du \Rightarrow I = \int \frac{du}{u} = \ln|u| + c = \ln|\sec\omega + \tan\omega| + c$

— exercise —

compute $\int \csc x$

try -1- $\int \frac{\sin x}{\sin^2 x} dx = I$ let $u = \cos x$, $du = -\sin x \, dx$ $\Rightarrow I = \int \frac{-du}{1-u^2} = -\frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du$

$= \frac{-1}{2}\left(-\ln|1-u| + \ln|1+u|\right) + c$

$\ln \frac{1+\cos t}{1-\cos t}$

$= -\frac{1}{2}\left(\ln|1-\cos x| + \ln|1+\cos x|\right) + c$

try -2- method in the prev. example

$I = \int \frac{\csc x(\cot x - \csc x)}{\cot x - \csc x} dx$ let $u = \cot x - \csc x$, $du = -\csc^2 x + (-\csc x \cot x)\, dx$

$\Rightarrow I = \int \frac{du}{u} = \ln|u| + c = \ln|\cot x - \csc x| + c$

Thm: (Integrals of Symmetric Functions)

Suppose f is cont on $[-a,a]$

Then

(i) if f is odd $\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$

(ii) if f is even $\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

Proof of ii

Let $u=-x \Rightarrow \begin{cases} du = -dx \\ x=-a \Rightarrow u=a \\ x=a \Rightarrow u=0 \end{cases}$

Now observe,

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-u)\,du = -\int_{a}^{0} f(u)\,du = \int_{0}^{a} f(u)\,du \overset{\text{since integral variables are dummy}}{=} \int_{0}^{a} f(x)\,dx$$

So, $$\int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{0} f(x)\,dx}_{\int_0^a f(x)dx} + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad \blacksquare$$

example: Compute $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{x^2+\sin x}{1+x^6} + \frac{114}{\pi}\right)dx$

Soln: Observe $\frac{x^2+\sin x}{1+x^6}$ is odd (Varify by showing $f(x) = -f(-x)$)

So, last thm implies (addition property of integrals)

$$I = \underbrace{\int_{-\pi/2}^{\pi/2} \frac{x^2+\sin x}{1+x^6}\,dx}_{0} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{114}{\pi}\,dx = \frac{114}{\pi}(x)\Big|_{-\pi/2}^{\pi/2} = \frac{114}{\pi}\pi = 114$$

Tecniques of Integration

Integration By Parts (IP)

Recall: The product rule for differantiation:

$$[f(x)\cdot g(x)]' = f'(x)g(x) + f(x)g'(x)$$

So, by integrating both sides gives,

$$\int [f(x)g(x)]'\,dx = \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

$$= \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx = f(x)g(x)$$

$$\Rightarrow \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$ "the IP formula"

By letting $u = f(x)$, $v = g(x)$ we get a form which is easier to remember:

$$\int u\,dv = uv - \int v\,du$$ IP for indefinite integrals formulae.

similarly $$\int_{x=a}^{x=b} u\,dv = uv - \int_{x=a}^{x=b} v\,du$$ IP formula for definite int.

Strategy for IP: If $\int u\,dv$ is very hard (or impossible) to compute, then compute $\int v\,du$ and use above IP formula.

example Evaluate the given integral.

① $I = \int \underset{u}{x}\, \underset{dv}{\sin x\,dx}$

Soln

$u = x \qquad du = dx$

$dv = \sin x\,dx \Rightarrow v = -\cos x$

$\Rightarrow I \overset{(\text{by IP})}{=} -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C$

② $I = \int x\sec^2 x\,dx$

Soln

$u = x \qquad du = dx$

$dv = \sec^2 x\,dx \Rightarrow v = \tan x$

$\Rightarrow I \overset{(\text{by IP})}{=} x\tan x - \int \tan x\,dx = x\tan x - (-\ln|\cos x|) + C$

$= x\tan x + \ln|\cos x| + C$

exercise: take the deriv. of this

(3) $I = \int_1^e x^3 \ln x \, dx$

Soln

$$u = \ln x \Rightarrow du = \frac{dx}{x}$$
$$dv = x^3 dx \Rightarrow v = \frac{x^4}{4} \underset{(\text{by IP})}{\Rightarrow} I = \frac{\ln x . x^4}{4}\Bigg|_{x=1}^{x=e} - \int_{1=x}^{e=x} \frac{x^4}{4} \cdot \frac{1}{x} dx$$

$$= \frac{\ln x \, x^4}{4}\Bigg|_{x=1}^{x=e} - \frac{x^4}{16}\Bigg|_{x=1}^{x=e} = \frac{e^4}{4} - \frac{e^4}{16} - \frac{1}{16}$$

(4) $I = \int \underbrace{\ln(x)}_{u} \underbrace{dx}_{dv}$

Solution

$$u = \ln x \Rightarrow du = dx . \frac{1}{x}$$
$$dv = dx \Rightarrow v = x \Rightarrow \binom{\text{by}}{\text{IP}} I = \ln x \, x - \int \frac{x \, dx}{x} = x \ln x - x + C$$

(5) $I = \int t^2 e^{-3t} dt$

Solution

$$u = t^2 \Rightarrow du = 2t\,dt$$
$$v = e^{-3t} dt \Rightarrow v = -\frac{1}{3} e^{-3t} \underset{\text{IP}}{\overset{\text{by}}{\Rightarrow}} I = t^2 . -\frac{1}{3} e^{-3t} - \underbrace{\int -\frac{1}{3} e^{-3t} \, 2t\,dt}_{(*)}$$

$$(*) \ \frac{2}{3} \int e^{-3t} \, 2t \, dt \quad \text{let } u = t, \ dv = e^{-3t} dt \Rightarrow du = dt, \ v = -\frac{1}{3} e^{-3t} \quad \text{by IP } (*) = t . -\frac{1}{3} e^{-3t} - \int -\frac{1}{3} e^{-3t} dt = \frac{-t e^{-3t}}{3} - \frac{e^{-3t}}{-9} + C_0$$

$$\text{So, } I = \frac{-t^2 . e^{-3t}}{3} + \left(-\frac{t . e^{-3t}}{3}\right) + \frac{e^{-3t}}{9} + C \qquad \left(C = \frac{2}{3} C_0\right)$$

— 16.02.2023 —

Recall **Integration by Parts (IP)**

$$\int u\,dv = uv - \int v\,du$$

$$\int_{x=a}^{x=b} u\,dv = uv\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du$$

example: $I = \int \underbrace{e^x}_{u} \underbrace{\sin x\,dx}_{dv}$

Solution

Let $u = e^x \Rightarrow du = e^x dx$

$dv = \sin x\,dx \Rightarrow v = -\cos x$

Then by IP $\quad I = e^x(-\cos x) - \int (-\cos x)\, e^x dx = -e^x\cos x + \underbrace{\int e^x \cos x\,dx}_{J} \quad (*)$

for J: let $\left.\begin{array}{l} u_0 = e^x \\ dv_0 = \cos x\,dx \end{array}\right\} \Rightarrow \begin{array}{l} du_0 = e^x dx \\ v_0 = \sin x \end{array}$

$\left(\text{by IP}\right)\; J = e^x \sin x - \underbrace{\int \sin x\, e^x dx}_{I} \Rightarrow J = e^x \sin x - I$

$\underset{(*)}{\Rightarrow} -e^x\cos x + e^x \sin x - I = I \Rightarrow I = \dfrac{e^x(\sin x - \cos x)}{2} + C$

exercise Compute $\int e^{ax} \sin(bx)\,dx$ where $a, b \in \mathbb{R}$ any constants.

TRIGONOMETRIC INTEGRALS

$(m, n \in \mathbb{Z}^+)$

- **Strategy** for $\int \sin^m x \cos^n x \, dx$

(a) n is odd

(we'll reserve 1 cos to use it in the substitution step)

* save 1 cosine
* use $\cos^2 x = 1 - \sin^2 x$
* let $u = \sin x$

(b) m is odd

* save 1 sine
* use $\sin^2 x = 1 - \cos^2 x$
* Let $u = \cos x$

(c) m, n are even

use half angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

example: compute $I = \int \sin^6 x \cos^3 x \, dx$

Solution: Here $m = 6$, $n = 3$ (odd). So, (a) implies

$$I = \int \sin^6 x \underbrace{\cos^2 x}_{1-\sin^2 x} \cos x \, dx \quad , \quad \text{Let } u = \sin x. \quad du = \cos x \, dx$$

$$\Rightarrow I = \int u^6 . (1 - u^2) \, du = \int (u^6 - u^8) \, du = \frac{u^7}{7} - \frac{u^9}{9} + C$$

back subst. $(u = \sin x)$ $\boxed{\frac{\sin^7(x)}{7} - \frac{\sin^9(x)}{9} + C}$

exercise

Compute $\int \sin^2 x \cos^2 x \, dx$ (Apply (c))

- **Strategy** for $\int \tan^m x \sec^n x \, dx$

(a) n is even $\Rightarrow$
- $*$ Save one $\sec^2 x$ factor
- $*$ use $\sec^2 x = 1 + \tan^2 x$
- $*$ let $u = \tan x$

(b) m is odd
- $*$ Save one $\sec x \tan x$ factor
- $*$ use $\tan^2 x = \sec^2 x - 1$
- $*$ let $u = \sec x$

(c) otherwise *We are all alone ...*

- manipulations
- reduction tecniques
- IP
- ...

might be usefull.

example compute $I = \int \frac{\tan^3 x}{\cos^5 x} \, dx$

soln: $I = \int \tan^3 x \sec^5 x \, dx \, dx$

Here $m = 3$, odd. (b) applies.

$I = \int \tan^2 x \sec^4 x \tan x \sec x \, dx$ let $u = \sec x$

$du = \tan x \sec x \, dx$

then $I = \int (\sec^2 x - 1) \sec^4 x \tan x \sec x \, dx = \int (u^2 - 1) u^4 \, du = \int (u^6 - u^4) \, dx = \frac{u^7}{7} - \frac{u^5}{5} + C$

$u = \sec x \Rightarrow$, $I = \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$

- **Strategy** for $\int \sin(mx)\cos(nx)\,dx$ or $\int \sin(mx)\sin(nx)\,dx$

or $\int \cos(mx)\cos(nx)\,dx$

Use the following identities:

a) $\sin\alpha \cdot \cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$

b) $\sin\alpha \cdot \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$

c) $\cos\alpha \cdot \cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$

example: Compute $I = \int_{-\pi}^{\pi/6} \sin(4x)\sin(5x)\,dx$

solution Use (b) with $\alpha = 4x$ $\beta = 5x$

$$I = \frac{1}{2}\int_{-\pi}^{\pi/6}\left(\cos(4x-5x)-\cos(4x+5x)\right)dx = \frac{1}{2}\int_{-\pi}^{\pi/6}\left[\cos(-x)-\cos(9x)\right]dx$$

$$= \frac{1}{2}\left(\sin(x)-\frac{1}{9}\sin(9x)\right)\Big|_{-\pi}^{\pi/6} = \frac{1}{2}\left[\left(\frac{1}{2}-\frac{1}{9}\underbrace{\sin\left(\tfrac{3\pi}{2}\right)}_{-1}\right)-\left(0-\frac{1}{9}(0)\right)\right]$$

$$= \frac{11}{36}$$

example Compute $I = \int \frac{\sin x}{\sin 2x}\,dx$

soln

$$I = \int \frac{\cancel{\sin x}}{2\cancel{\sin x}\cos x}\,dx = \frac{1}{2}\int \sec x\,dx \text{ (as before) } = \frac{1}{2}\left(\ln|\sec x+\tan x|\right)+c$$

example Compute $I = \int_{\frac{\pi}{3}}^{\pi} \frac{dx}{\cos x-1}$

soln

Use the identity $\cos x = 1-2\sin^2\left(\frac{x}{2}\right)$

$$I = \int_{\frac{\pi}{3}}^{\pi}\frac{dx}{\cancel{1}-2\sin^2\left(\frac{x}{2}\right)\cancel{-1}} = \frac{-1}{2}\int_{\pi/3}^{\pi}\csc^2\left(\frac{x}{2}\right)dx$$

let $u = \cot\left(\frac{x}{2}\right)$

$du = -\csc^2\left(\frac{x}{2}\right)\cdot\frac{1}{2}$

$$I = \int_{x=\frac{\pi}{3}}^{x=\pi} du = u\Big|_{x=\pi/3}^{x=\pi}$$

$$I = \cot\left(\frac{x}{2}\right)\Big|_{\pi/3}^{\pi} = \cot\left(\frac{\pi}{2}\right)-\cot\left(\frac{\pi}{6}\right)$$

$$= 0-\sqrt{3} = -\sqrt{3}$$

REDUCTION FORMULAS

$(n \in \mathbb{Z}^+)$

(1) $$\int \sin^n x\,dx = \frac{-1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

(2) $$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx$$

(3) $$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n+2}{n-1}\int \sec^{n-2}x\,dx$$

Similarly, cos, cot, csc ... (do the pf's as exercise)

proof of (1)

we apply Integration by Parts.

Let $u = \sin^{n-1}x$ $\quad\Rightarrow\quad$ $du = (n-1)\sin^{n-2}(x)\cos(x)\,dx$

$dv = \sin x\,dx$ (saved 1 sine) $\qquad v = -\cos x$

$$\text{So, } I = \int \sin^n x\,dx = -\sin^{n-1}x\cos x - \int -\cos x\,(n-1)\sin^{n-2}x\cos x\,dx$$

$$-\cos^2 x = -1 + \sin^2 x$$

$$\text{thus } I = -\sin^{n-1}\cos x - (n-1)\underbrace{\int (\sin^2 x - 1)\sin^{n-2}x\,dx}_{\int(\sin^n x - \sin^{n-2}x)\,dx}$$

$$I = -\sin^{n-1}\cos x - (n-1)\underbrace{\int \sin^n x\,dx}_{I} + (n-1)\int \sin^{n-2}x\,dx$$

$$I\,(1+n-1) = -\cos x\sin^{n-1} - (n-1)(\int \sin^{n-2}x\,dx$$

∴ dividing both sides with n gives the required formulas.

INTEGRALS OF RATIONAL FUNCTIONS

Goal: Given two polynomials

$$P(x) = \underset{\neq 0}{a_n} x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$$Q(x) = \underset{\neq 0}{b_m} x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0$$

We want to compute $\int \frac{P(x)}{Q(x)} dx$.

__Remark:__ $\frac{P(x)}{Q(x)}$ is called a rational function.

This can be done by using long division (if necessary) and then applying the partial fraction method.

~Steps~

(I) If $n = \deg(P) \geq \deg(Q) = m$

then first divide $P(x)$ by $Q(x)$ using the long division.

$$\frac{P(x)}{Q(x)} = \underbrace{S(x)}_{\text{resulting polyn.}} + \frac{\overbrace{R(x)}^{\text{remainder polyn}}}{Q(x)}$$

* this division should continue until $\deg(R) < \deg(Q)$ ($\neq$)

(II) $$\int \frac{P(x)}{Q(x)} dx = \int S(x)\,dx + \underbrace{\int \frac{R(x)}{Q(x)} dx}_{\text{part. frac. method}}$$

(III) Rules for partial fractions

* Factor $Q(x)$ as much as possible
* Use the following table to determine partial fractions needed

(continued next page)

	factor of $Q(x)$	corresponding partial fraction contribution
linear factor	$(ax+b)^r$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_r}{(ax+b)^r}$
irreducable factor	$(ax^2+bx+c)^r$ $\Delta = b^2-4ac < 0$	$\frac{A_1(x)+B_1}{ax^2+bx+c} + \frac{A_2(x)+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_r(x)+B_r}{(ax^2+bx+c)^r}$

(r: multiplicity)

- once all unknowns $A_1, \dots, A_r, B_1, \dots, B_r$ are determined we can easily compute

$$\int \frac{R(x)}{Q(x)}\,dx$$

examples Compute each integral.

① $I = \int \frac{x^3+2x^2-2x+1}{x^2-1}\,dx$

solution $\deg(P) = 3 > \deg(Q) = 2$. So, start with long division

$$\begin{array}{r|l} x^3+2x^2-2x+1 & x^2-1 \\ \hline -x^3+x & x+2 \\ \hline 2x^2-x+1 & \\ -2x^2+2 & \\ \hline -x+3 & \end{array}$$

$-x+3 \rightarrow \deg < 2$

So, $\frac{x^3+2x^2-2x+1}{x^2-1} = \underbrace{x+2}_{S(x)} + \frac{\overset{R(x)}{3-x}}{\underset{Q(x)}{x^2-1}}$

Now, $Q(x) = (x-1)(x+1) = x^2-1$

$$\frac{3-x}{(x^2-1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} \Rightarrow A(x)+A+B(x)-B = 3-x \qquad \begin{array}{l} A+B=-1 \\ A-B=3 \end{array} \Rightarrow A=1 \;\; B=-2$$

So, $\frac{3-x}{x^2-1} = \frac{1}{x-1} + \frac{-2}{x+1}$. Hence I becomes $I = \int \left(x+2+\frac{1}{x-1}+\frac{-2}{x+1}\right)dx$

$$I = \frac{x^2}{2} + 2x + \ln|x-1| - 2\ln|x+1| + C$$

(2) $I = \int \dfrac{-12x}{4x^2-4x+1}\,dx$

Soln $\deg P < 2$. No long division needed.

$q(x) = 4x^2-4x+1 = (2x-1)^2$ we have 1 distinct factor with multiplicity $r=2$

$$\frac{-12x}{(2x-1)^2} = \frac{A}{\underset{(2x-1)}{(2x-1)}} + \frac{B}{(2x-1)^2} \Rightarrow 2Ax - A + B = -12x \Rightarrow \begin{array}{l} B = A \\ 2A = -12 \\ A = -6 = B \end{array}$$

So, $\dfrac{-12x}{(2x-1)^2} = \dfrac{-6}{2x-1} - \dfrac{6}{(2x-1)^2}$

Hence $I = -6\int\left(\dfrac{1}{2x-1} + \dfrac{1}{(2x-1)^2}\right)dx = -6\left(\dfrac{\ln|2x-1|}{2} - \dfrac{(2x-1)^{-1}}{2}\right) + C$

$$= -3\ln|2x-1| + \frac{3}{(2x-1)} + C$$

(3) $I = \int \dfrac{x^4+81}{x(x^2+9)^2}\,dx$

$\deg(P) = 4 < \deg(Q) = 5$ no long division needed.

$q(x) = x\cdot(x^2+9)^2$ 2 distinct factors. $\nearrow x$ mult. $=1$ (linear factor)
$\searrow (x^2+9)$ mult. $=2$ (irreducable $\Delta<0$) ($a=1$ $b=0$ $c=9$)

$$\frac{x^4+81}{x(x^2+9)^2} = \frac{A}{\underset{(x^2+9)^2}{x}} + \frac{B_1(x)+C_1}{\underset{(x^3+9x)}{x^2+9}} + \frac{B_2(x)+C_2}{\underset{(x)}{(x^2+9)^2}}$$

$\Rightarrow \underline{Ax^4} + 18x^2A + \underset{A=1}{81A} + \underline{B_1x^4} + \underset{C_1=0}{C_1x^3} + 9B_1x^2 + 9C_1x + B_2x^2 + C_2x$

$A + B_1 = 1 \Rightarrow B_1 = 0$
$x^2(18A + 9B_1 + B_2) = 0$
$x(\underset{0}{9C_1} + \underset{0}{C_2}) = 0$
$B_2 = -18$

$$\frac{x^4+81}{x(x^2+9)^2} = \frac{1}{x} + \frac{-18x}{(x^2+9)^2}$$

$I = \int \dfrac{1}{x}\,dx + \int \dfrac{-18x}{(x^2+9)^2}\,dx$

Let $x^2+9 = u$
$2x\,dx = du$
So this integral becomes
$\int \dfrac{-9\,du}{u^2} = 9u^{-1} + C$
$= 9(x^2+9)^{-1} + C$

overall, $\boxed{I = \ln|x| + 9(x^2+9)^{-1} + C}$

INVERSE TRIGONOMETRIC SUBSTITUTIONS

Strategy-

expression appeared in the integral	corresponding substitution (suggested)
$a^2 - x^2$	$x = a\sin\theta$
$x^2 - a^2$	$x = a\sec\theta$
$a^2 + x^2$	$x = a\tan\theta$

Example compute each integral

① $I = \displaystyle\int \frac{x\,dx}{\sqrt{9-x^2}}$

- instead of this new tecnique basic u subst. still applies here

let $u = 9 - x^2$ $\Rightarrow$ $I = \displaystyle\int \frac{-1}{2}\frac{du}{\sqrt{u}} = \frac{-1}{2}\,2u^{\frac{1}{2}} + c = -\sqrt{u} + c = -\sqrt{9-x^2} + c$

$du = -2x\,dx$

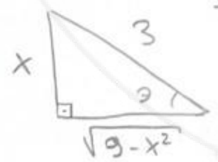
- using the new idea

let $x = 3\sin\theta$ $\quad I = \displaystyle\int \frac{3\sin\theta \cdot 3\cos\theta\, d\theta}{\sqrt{\underbrace{9 - 9\sin^2\theta}_{9(1-\sin^2\theta)}}} = \int \frac{9\sin\theta\cos\theta}{3\cos\theta}\,d\theta$

$dx = 3\cos\theta\, d\theta$

$= \displaystyle\int 3\sin\theta\, d\theta = -3\cos\theta + c.$

$\sin\theta = \dfrac{x}{3}$



$\cos\theta = \dfrac{\sqrt{9-x^2}}{3}$

$\boxed{\text{back subst.}} = \dfrac{-3\sqrt{9-x^2}}{3} + c = -\sqrt{9-x^2} + C$

(2) $$I = \int \frac{\sqrt{x^2-49}}{x}\,dx$$

let $x = 7\sec\theta$

$dx = 7\sec\theta\tan\theta\,d\theta$

$$I = \int \frac{\sqrt{49(\sec^2\theta-1)}}{\cancel{7\sec\theta}}\,\cancel{7\sec\theta}\tan\theta\,d\theta$$

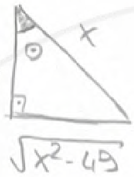
(consider $\tan\theta > 0$)

$$= \int 7\tan\theta\tan\theta\,d\theta = 7\int \tan^2\theta\,d\theta = 7\int(\sec^2\theta-1)\,d\theta$$

$$= 7\tan\theta + \theta + C$$

$\sec\theta = \frac{x}{7}$

$\cos\theta = \frac{7}{x}$



back subst.
$$= \frac{7\sqrt{x^2-49}}{7} + 7\operatorname{arcsec}\left(\frac{x}{7}\right) + C$$

example

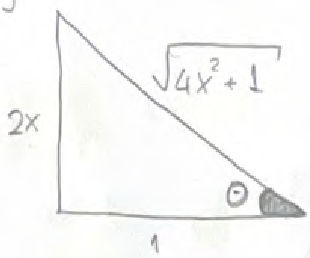
Compute $I = \int \frac{8}{(4x^2+1)^2} dx$

<u>soln</u>: let $2x = \tan\theta$

$dx = \frac{1}{2}\sec^2\theta \, d\theta$ $\Rightarrow I = \int \frac{8}{(\underbrace{\tan^2\theta+1}_{\sec\theta})^2} \frac{1}{2}\sec^2\theta \, d\theta$

$= 4\int \frac{1}{\sec^2\theta} d\theta = 4\int \cos^2\theta \, d\theta$

$\int 4\cos^2\theta \, d\theta = 4\int \frac{1}{2}\left(\cos 2\theta + \frac{1}{2}\right) d\theta = 2\int (\cos 2\theta + 1) d\theta$ ⊛



$\Rightarrow \sin\theta = \frac{2x}{\sqrt{4x^2+1}}$ $\cos\theta = \frac{1}{\sqrt{4x^2+1}}$

⊛: $2\int (\cos 2\theta + 1) \, d\theta = 2 \cdot \left[\frac{1}{2}(\sin 2\theta) + \theta\right] + C$

back substitution $= 2\left[\frac{2x}{4x^2+1} + \tan^{-1}(2x)\right] + C$

$\tan\left(\frac{x}{2}\right)$ substitution

This is useful when we integrate rational expressions involving trigonometric functions.

idea: By letting $t = \tan\left(\frac{x}{2}\right)$ we can easily compute $dx, \sin x, \cos x$, $\sin\left(\frac{x}{2}\right), \cos\left(\frac{x}{2}\right), \sin(2x), \cos(2x) \dots$

More importantly this substitution transforms the given integral into the integral of a rational function in the new t variable.

More precisely: Let $t = \tan\left(\frac{x}{2}\right)$ (where $-\pi < x < \pi$)

$$dt = \sec^2\left(\frac{x}{2}\right) \frac{1}{2} dx$$

example

Find $I = \int \frac{1}{\sin x} dx$

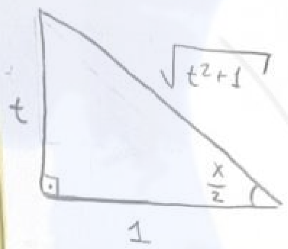
Soln

Let $t = \tan\left(\frac{x}{2}\right)$ Then by the formulas we have

$$I = \int \frac{1}{\cancel{\frac{2t}{1+t^2}}} \cancel{\frac{2}{1+t^2}} dt = \int \frac{dt}{t}$$

$$= \ln|t| + c$$

back subst. $= \ln\left|\tan\left(\frac{x}{2}\right)\right| + c$



$\Rightarrow \cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}}$

$\Rightarrow \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}$

$\Rightarrow \sec^2\left(\frac{x}{2}\right) = \frac{1}{\cos^2\left(\frac{x}{2}\right)} = 1+t^2$

$\Rightarrow dt = (1+t^2)\frac{1}{2} dx \Rightarrow dx = \frac{2\,dt}{1+t^2}$

Also, $\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

$\Rightarrow \sin(x) = \frac{2t}{1+t^2}$

Similarly, $\cos(x), \sin(2x), \cos(2x) \dots$ can be calculated.

example: Compute $I = \int \frac{dx}{\sin x - \cos x}$

Solution Let $t = \tan\left(\frac{x}{2}\right)$

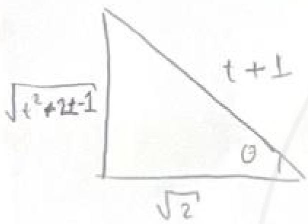
by previous formulas $\quad I = \int \dfrac{\frac{2}{\cancel{1+t^2}}}{\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}}\, dt = \int \frac{2}{(t+1)^2 - 2}\, dt$

Let $t+1 = \sqrt{2}\sec\theta \Rightarrow dt = \sec\theta \tan\theta \sqrt{2}\, d\theta$

$$I = \int \frac{\cancel{2}}{\underbrace{2\sec^2\theta - 2}_{\cancel{2}\tan^2\theta}} \sqrt{2}\sec\theta\, \cancel{\tan\theta}\, d\theta = \int \frac{\sqrt{2}}{\cos\theta} \frac{\cos\theta}{\sin\theta}\, d\theta = \sqrt{2}\int \csc\theta\, d\theta$$

earlier result $\quad = -\ln|\csc\theta + \cot\theta| \cdot \sqrt{2} + C$

back subst



$$= -\sqrt{2}\ln\left|\frac{t+1}{\sqrt{t^2+2t-1}} + \frac{\sqrt{2}}{\sqrt{t^2+2t-1}}\right| + C$$

$$= -\sqrt{2}\ln\left|\frac{\tan\left(\frac{x}{2}\right) + 1 + \sqrt{2}}{\sqrt{\tan^2\left(\frac{x}{2}\right) + 2\tan\left(\frac{x}{2}\right) - 1}}\right| + C$$

Some other substitutions

① Compute $I = \int \frac{1}{1+\sqrt{2x}}\, dx$

soln

Let $\sqrt{2x} = u$

$\frac{1}{2\sqrt{2x}}\, 2\, dx = du \Rightarrow dx = u\, du$

$$\Rightarrow I = \int \frac{u}{1+u}\, du = \int \frac{u+1-1}{u+1}\, du = \int \left(1 - \frac{1}{u+1}\right) du = u - \ln|u+1| + C$$

back subst $\quad = \sqrt{2x} - \ln|\sqrt{2x} + 1| + C$

② Compute $I = \int \frac{1}{\sqrt{x}\,(1+x^{1/3})}\,dx$

Solution

Let $x = u^6$ $(u>0)$ (equivalently we have $\sqrt[6]{x} = u$)

$\Rightarrow dx = 6u^5\,du$

$$\Rightarrow I = 6\int \frac{u^{5}\,du}{u^3(1+u^2)} = 6\int \frac{u^2\,du}{1+u^2} = 6\int \frac{u^2+1-1}{u^2+1}\,du = 6\int\left(1 - \frac{1}{u^2+1}\right)du$$

$$= 6\left(u + \tan^{-1}(u)\right) + c$$

back substitution $= 6\left[\sqrt[6]{x} - \tan^{-1}\left(\sqrt[6]{x}\right)\right] + c$

Improper Integrals

So far we have seen only proper definite integrals.

(that is the integrals whose integrands are bounded continuous functions and whose integration regions are finite closed intervals)

Now, we would like to study some other type of definite integrals. (which we call "improper" definite integrals) by weakening the above conditions of proper integrals. They will arise as certain limits of proper definite integrals.

Def^n: A definite integral $\int_a^b f(x)\,dx$ is called "improper integral" if

- either $a = \pm\infty$ or $b = \pm\infty$ (Type-1)

  or

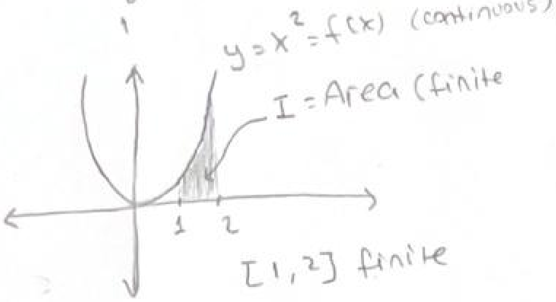
- $f(x)$ has some infinite discontinuity at some point(s) in the finite interval $[a,b]$. (Type-2)

If both happen we call it mixed type improper integral

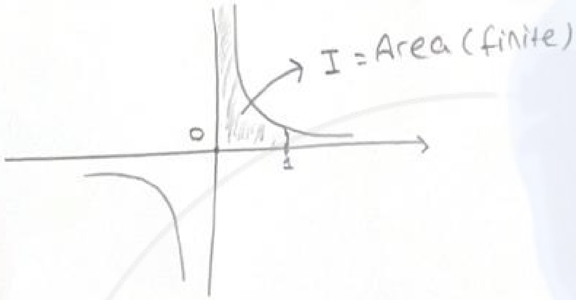
example:

$I = \int_1^2 x^2\,dx$ is proper.

$I = \int_1^\infty x^2\,dx$ is improper



$I = \int_0^1 \frac{1}{\sqrt{x}}\,dx$ is improper (type-2)



Def^n (Type-I Improper Integrals)

(i) $\int_a^\infty f(x)\,dx \overset{\text{defn}}{=} \lim_{t\to\infty} \int_a^t f(x)\,dx$ (provided limit exists)

(ii) $\int_{-\infty}^b f(x)\,dx \overset{\text{defn}}{=} \lim_{t\to-\infty} \int_t^b f(x)\,dx$ (provided limit exists)

(iii) $\int_a^\infty f(x)\,dx$ & $\int_{-\infty}^b f(x)\,dx$ are called convergent if the corresponding limit exist. Otherwise we simply say they are divergent.

(or they do not exist (DNE))

(iv) If $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are both convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx \overset{\text{defn}}{=} \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$ which is also convergent.

Def^n (Type-II Improper Integrals)

(i) Suppose f is cont. on $[a,b)$ and has ∞-discontinuity at b

Then $\displaystyle\int_a^b f(x)\,dx \overset{\text{def}^n}{=} \lim_{t\to b^-} \int_a^t f(x)\,dx$ (provided limit exists)

(ii) Suppose f is cont. on $(a,b]$ and has ∞-discont. at a

then $\displaystyle\int_a^b f(x)\,dx \overset{\text{def}^n}{=} \lim_{t\to a^+} \int_t^b f(x)\,dx$ (provided limit exists)

(iii) The improper integral $\displaystyle\int_a^b f(x)\,dx$ is called **convergent** if the defining limit exist

Otherwise, is is called **divergent**.

(iv) If f has ∞-discont. at c where $a<c<b$ and if f is cont. on $[a,c)\cup(c,b]$, and both $\displaystyle\int_a^c f(x)\,dx$ and $\displaystyle\int_c^b f(x)\,dx$ are **convergent**

Then

$$\int_a^b f(x)\,dx \overset{\text{def}^n}{=} \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

which is also **convergent**.

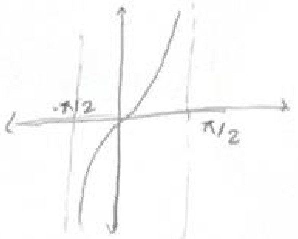
- If $\displaystyle\int_a^c f(x)\,dx$ or $\displaystyle\int_c^b f(x)\,dx$ is **divergent**, then $\displaystyle\int_a^b f(x)\,dx$ is **divergent**.

example Evaluate each integral if it exists.

① $I = \int_0^\infty \frac{1}{x^2+1}\,dx$ (type-1)

soln

$I = \lim_{t\to\infty} \int_0^t \frac{1}{x^2+1}\,dx$



$I = \lim_{t\to\infty} \tan^{-1}(x)\Big|_0^t = \lim_{t\to\infty}\left[\tan^{-1}(t) - 0\right]$

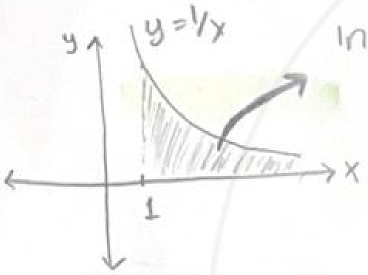
$= \frac{\pi}{2}$ ∴ I is convergent

② $I = \int_1^\infty \frac{1}{x}\,dx$

soln

$I = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty}\left[\ln|t| - 0\right] = \infty$ (DNE)

∴ I is divergent

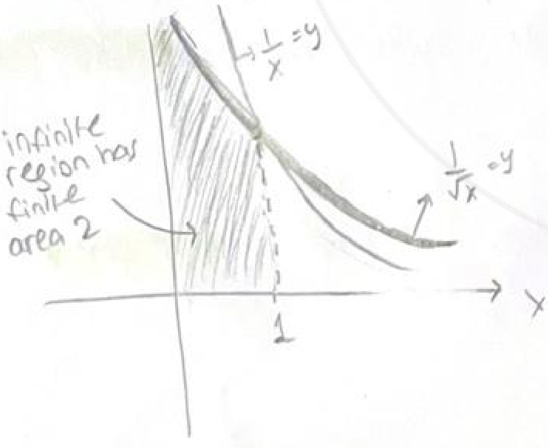


Infinite region has infinite area (i.e. its area d.n.e)

③ $I = \int_0^1 \frac{dx}{\sqrt{x}}$ (type-II)

soln

$I = \lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} 2\sqrt{x}\Big|_t^1 = \lim_{t\to 0^+} 2(1-\sqrt{t}) = 2$ ∴ I is conv. I converges to 2.



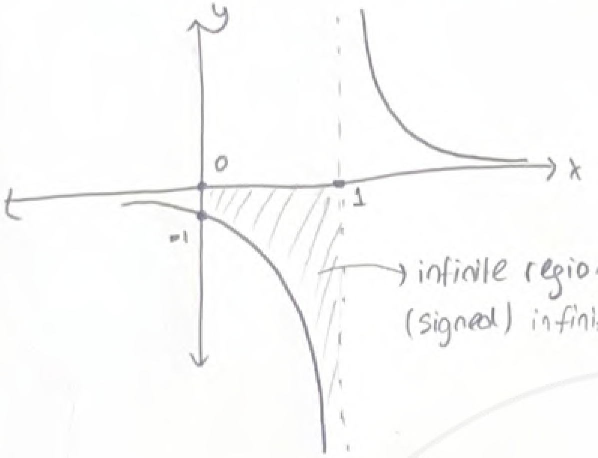
~ exercise ~

$\int_0^1 \frac{1}{x}\,dx = \infty$ (DNE) divergent try-1 $\lim_{t\to 0^+} \frac{\ln|1|}{0} - \frac{\ln|t|}{\infty}$ → dne

(4) $I = \int_0^1 \frac{dx}{x-1}$ (Type-II)

Soln.

$$I = \lim_{t \to 1^-} \ln|x-1| \Big|_0^t = \lim_{t \to 1^-} \left[\ln|\underbrace{t-1}_{\to 0}| - 0 \right] = -\infty \text{ d.n.e}$$



infinite region has (signed) infinite area $-\infty$

example: Determine if $I = \int_0^\infty \sin x \cdot e^{-x} dx$ is convergent.

Soln. This is Type-I

we know $0 \leq$ - This is ACT example. I stopped writing. It's on the next pages example Evaluate if it exists.

(1) $I = \int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$ (mixed type)

Soln: let $I_1 = \int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ problem occurs for $x=1$. Let's start there. $I_1 = \lim_{t\to 1^+} \int_t^2 \frac{1}{|x|\sqrt{x^2-1}}\,dx$

$$= \lim_{t\to 1^+} \sec^{-1}(x)\Big|_t^2 = \lim_{t\to 1^+} (\underbrace{\sec^{-1}(2)}_{\pi/3} - \sec^{-1}(t))$$

$$\boxed{= \frac{\pi}{3} \text{ (exists)}}$$

let $I_2 = \int_2^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{t\to\infty} \int_2^t \frac{dx}{x\sqrt{x^2-1}} = \lim_{t\to\infty} \sec^{-1}(x)\Big|_2^t = \lim_{t\to\infty} (\sec^{-1}(t) - \sec^{-1}(2))$

$$= \frac{\pi}{2} - \frac{\pi}{3} = \boxed{\frac{\pi}{6} \text{ (exists)}}$$

Hence, I exists. $I = I_1 + I_2 = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$

(2) $\int_0^2 \frac{dx}{\sqrt{|x-1|}} = I$

soln: let $I_1 = \int_0^1 \frac{dx}{\sqrt{|x-1|}}$ & $I_2 = \int_1^2 \frac{dx}{\sqrt{|x-1|}}$

$$I_1 = \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt{1-x}} \overset{\binom{u=1-x}{du=-dx}}{=} \lim_{t\to 1^-} \frac{-(1-x)^{1/2}}{1/2}\Bigg|_0^t = \lim_{t\to 1^-} -2(\sqrt{1-t} - 1) = 2 = I_1$$

$$I_2 = \lim_{t\to 1^+} \int_1^2 \frac{dx}{\sqrt{x-1}} \overset{\binom{u=x-1}{du=dx}}{=} \lim_{t\to 1^+} \frac{(x-1)^{1/2}}{1/2}\Bigg|_t^2 = 2 = I_1$$

$I = I_1 + I_2$, $2+2 = 4 = I$ //

P-test for improper integrals

Suppose $a > 0$. Then

(i) $$\int_0^a \frac{1}{x^p}\,dx = \begin{cases} \text{convergent} & \text{if } p < 1 \\ \text{divergent} & \text{if } p \geq 1 \end{cases}$$

(ii) $$\int_a^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{convergent if} & p > 1 \\ \text{divergent if} & p \leq 1 \end{cases}$$

(**Proof of ii**) first one is exercise

$$\int_a^\infty \frac{1}{x^p}\,dx \overset{\text{def}^n}{=} \lim_{t\to\infty} \int_a^t x^{-p}\,dx = \begin{cases} \lim\limits_{t\to\infty} \ln|x| \Big|_a^t & \text{if } p = 1 \\ \lim\limits_{t\to\infty} \dfrac{x^{-p+1}}{1-p} \Big|_a^t & \text{if } p \neq 1 \end{cases}$$

So $$\begin{cases} \lim\limits_{t\to\infty} (\overset{\infty}{\ln|t|} - \overbrace{\ln|a|}^{\text{finite}}) & \text{if } p = 1 \\ \lim\limits_{t\to\infty} \left(\dfrac{1}{t^{p-1}} - \dfrac{1}{a^{p-1}} \right) & \text{if } p \neq 1 \end{cases} = \begin{cases} \text{divergent} & \text{if } p \leq 1 \\ \text{convergent} & \text{if } p > 1 \end{cases}$$

if $p > 1$: 0 if $p < 1$: ∞

■

example Given $I = \int_0^1 \frac{dx}{x^{1/3}}$

(a) Determine convergence.

(b) Compute $J = \int_{-8}^1 \frac{dx}{x^{1/3}}$ if it exists.

Soln

- By p-test (with $p = 1/3$) I is convergent.
- Consider $J_1 = \int_{-\infty}^0 \frac{dx}{x^{1/3}}$ and $J_2 = \int_0^1 \frac{dx}{x^{1/3}} = I$

observe that, by def^n. J converges if J_2 & J_1 are convergent

$J_2 = I$ is convergent

for $J_2 (= I)$

$$\lim_{t\to 0^+} \int_t^1 x^{-1/3}\,dx = \lim_{t\to 0^+} \left[\frac{x^{2/3}}{2/3} \Big|_t^1 = \frac{3}{2}\left(1 - \underset{0}{t^{2/3}}\right) \right]$$

$J_2 = 3/2$

For J_1

$$\lim_{t\to 0^-} \int_{-8}^t x^{-\frac{1}{3}}\,dx = \lim_{t\to 0^-} \frac{3}{2} x^{2/3} \Big|_{-8}^t = \lim_{t\to 0^-} \frac{3}{2}\left(t^{2/3} - (-8)^{2/3}\right)$$

$J_1 = -4 \cdot \frac{3}{2} = -6$

$J = J_1 + J_2 = 2/3 - 6 = 16/3$ //

exercise

$$I = \int_2^{\infty} \frac{2}{x^2-1}\, dx \qquad (\text{Answer: } \ln 3)$$

(Direct) Comparison Test ((CT))

(a: fixed constant)

Suppose $0 \leq f(x) \leq g(x)$ $\forall x \in [a, \infty)$ (f, g are cont)

Then **(i)** $\int_a^{\infty} g(x)\,dx$ conver. $\Rightarrow \int_a^{\infty} f(x)\,dx$ conv.

(ii) $\int_a^{\infty} f(x)\,dx$ div. $\Rightarrow \int_a^{\infty} g(x)\,dx$ diver.

Proof of (i) ((ii) follows from (i) as being they are contropositive of each other)

Let $F(t) = \int_a^{t} f(x)\,dx$

$G(t) = \int_a^{t} g(x)\,dx$

Then by assumption, since $\int_a g(x)\,dx$ is convergent

$\lim\limits_{t \to \infty} G(t) = \int_a^{\infty} g(x)\,dx = L$ (finite) exists

In order to show $\int_a^{\infty} f(x)\,dx$ convergent, we need to show $\lim\limits_{t\to\infty} F(t)$ exists.

Observe that, F & G are both increasing because f & g are positive

Also, $F(t) \leq G(t)$ $\forall t \in [a, \infty)$

$$\Rightarrow \lim_{t\to\infty} F(t) \leq \lim_{t\to\infty} G(t) = L \text{ (finite)} \Rightarrow F(t) \leq L, \ \forall t \in [a, \infty)$$

Therefore, being increasing and bounded above (by L), we conclude that $\lim\limits_{t\to\infty} F(t)$ exists. ⊠

Properties of Convergent Improper Integrals

Suppose $\int_a^\infty f(x)\,dx$, $\int_a^\infty g(x)\,dx$ are both convergent,

and $c \in \mathbb{R}$ any constant. (f, g are cont.)

Then (i) $\int_a^\infty [f(x) \pm g(x)]\,dx = \int_a^\infty f(x)\,dx \pm \int_a^\infty g(x)\,dx$

(ii) $\int_a^\infty c f(x)\,dx = c\int_a^\infty f(x)\,dx$

proof of (ii)

By assumption, if $F(t) = \int_a^t f(x)\,dx$,

then $\lim_{t\to\infty} F(t) = L$ (exists | finite)

Now, $\int_a^\infty c f(x)\,dx = \lim_{t\to\infty} \int_a^t c f(x)\,dx = c \lim_{t\to\infty} \int_a^t f(x)\,dx = c \lim_{t\to\infty} F(t) = cL$

$= c\int_a^\infty f(x)\,dx$

Def^n: $\int_a^\infty f(x)\,dx$ is called **absolutely convergent**

if $\int_a^\infty |f(x)|\,dx$ is convergent

Thm: (Absolute Convergence Theorem) (ACT)

Absolutely converge $\Rightarrow$ conv.

[i.e. $\int_a^\infty |f(x)|\,dx$ conv. $\Rightarrow \int_a^\infty f(x)\,dx$ conv.]

proof By assumption $\int_a^\infty |f(x)|\,dx$ exists (finite)

we know $-|f(x)| \le f(x) \le |f(x)|$, $\forall x \in [a, \infty)$

$\Rightarrow 0 \le f(x) + |f(x)| \le 2|f(x)| \quad \forall x \in [a, \infty)$

$\xRightarrow[\text{property (ii)}]{} \int_a^\infty 2|f(x)|$ is convergent $\xRightarrow[\text{(CT)}]{} \int_a^\infty [f(x) + |f(x)|]\,dx$ is conv.

Remark All defn^s. and results about absolute convergence, and also Comparison Test and property of convergent improper integrals can be stated and verified for Type II improper integrals as well.

For instance,

CT for Type-II Improper Integrals

Suppose $0 \leq f(x) \leq g(x)$, $\forall x \in [a,b)$ (Here f, g are cont. on $[a,b)$, and they both have ∞-discont. at b)

Then (i) $\int_a^b g(x)\,dx$ conv. $\Rightarrow \int_a^b f(x)\,dx$ conv.

(ii) $\int_a^b f(x)\,dx$ div. $\Rightarrow \int_a^b g(x)\,dx$ div.

example Determine if $I = \int_a^\infty \sin x \, e^{-x}\,dx$ is convergent.

Soln: This is type-I ✓

we know $0 \leq |\sin(x)| \leq 1$ $\forall x \in [a,\infty)$

$\Rightarrow 0 \leq e^{-x}|\sin(x)| \leq e^{-x}$, $\forall x \in [a,\infty)$

Also, $\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_0^t e^{-x}\,dx = \lim_{t\to\infty} -e^{-x}\Big|_0^t = \lim_{t\to\infty} [-e^{-t} + \underset{1}{e^0}] = 0+1$ (finite)

$\Rightarrow \int_0^\infty e^{-x}\,dx$ conv. $\underset{C.T.}{\Longrightarrow} \int_0^\infty |\sin x| e^{-x}\,dx$ conv. $\underset{(A.C.T.)}{\Rightarrow} I$ is convergent.

Limit Comparison Test

Suppose $f(x) \geq 0$, $g(x) \geq 0$ on $[a,\infty)$

Consider $L = \lim_{x\to\infty} \frac{f(x)}{g(x)}$ (So, $0 \leq L \leq \infty$)

then (i) $L = 0$ & $\int_a^\infty g(x)\,dx$ conv. $\Rightarrow \int_a^\infty f(x)\,dx$ conv.

(ii) $L = \infty$ & $\int_a^\infty g(x)\,dx$ div. $\Rightarrow \int_a^\infty f(x)\,dx$ div.

(iii) $0 \underset{\neq}{<} \overset{\text{finite}}{L} \underset{\neq}{<} \infty \Longrightarrow \int_a^\infty g(x)\,dx$ convergent $\Leftrightarrow \int_a^\infty f(x)\,dx$ conv.; divergent $\Leftrightarrow \int_a^\infty f(x)\,dx$ div.

(i.e. they behave the same)

proof of (iii)

Assume $L \neq 0$ and finite (positive finite numbers)

Then $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L \Rightarrow \exists\, \varepsilon > 0$ such that

$$0 < L-\varepsilon < \frac{f(x)}{g(x)} < L+\varepsilon \quad \forall x \geq M \quad \text{for some } M > 0$$

$$\Longrightarrow \quad 0 < (L-\varepsilon)\, g(x) \underset{\text{I}}{<} f(x) \underset{\text{II}}{<} (L+\varepsilon)\, g(x) \quad \forall x \geq M$$

now observe $\int_a^\infty g(x)dx$ conv. $\Leftrightarrow (L+\varepsilon)\int_M^\infty g(x)dx$ conv. $\underset{\text{C.T.}}{\overset{\text{II}}{\Longrightarrow}} \int_M^\infty f(x)dx \Leftrightarrow \int_a^\infty f(x)dx$ conv.

$\int_a^\infty f(x)dx$ conv. $\Leftrightarrow \int_M^\infty f(x)dx$ conv. $\underset{\text{C.T}}{\overset{\text{I}}{\Longrightarrow}} (L-\varepsilon)\int_M^\infty f(x)dx$ conv. $\Leftrightarrow \int_a^\infty g(x)dx$ conv.

example Determine whether given integral is convergent or not

1. $I = \int_4^\infty \frac{1}{\sqrt{x}+1}\, dx$

Solution

Let $f(x) = \frac{1}{\sqrt{x}+1} > 0$ & $g(x)\ \frac{1}{\sqrt{x}} > 0$

observe $\lim\limits_{x\to\infty} \dfrac{\frac{1}{\sqrt{x}+1}}{\frac{1}{\sqrt{x}}} = \lim\limits_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}+1} \overset{(LR)}{=} 1$ (finite

Also, $\int_4^\infty \frac{1}{\sqrt{x}}\, dx$ is divergent by p-test $\left(p = \frac{1}{2} \leq 1\right)$

$\underset{\text{L.C.T. (iii)}}{\Rightarrow}$ I is also divergent

Remark: LCT has a version also for Type-II improper integrals as well

In that case, $L = \lim\limits_{\substack{x\to b^- \\ x\to a^+}} \frac{f(x)}{g(x)}$

(2) $$I = \int_2^\infty \frac{1}{\sqrt{x}\ln x}\,dx \qquad \text{(type-1)}$$

soln: Say $f(x) = \dfrac{1}{\sqrt{x}\ln x} > 0 \qquad g(x) = \dfrac{1}{x} > 0$

observe $L = \lim\limits_{x\to\infty} \dfrac{\frac{1}{\sqrt{x}\ln x}}{1/x} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{\ln x} \overset{LR}{=} \infty$

Also, $\int_2^\infty \frac{1}{x}\,dx$ is divergent by p-test ($p=1$)

$\underset{\text{LCT (ii)}}{\Rightarrow}$ I is

(3) $$I = \int_{1/2}^{1} \frac{\sin x}{\ln x}\,dx \qquad \text{(type-II)}$$

Solution consider $f(x) = \dfrac{-\sin x}{\ln x} > 0 \quad \forall x \in [1/2, 1)$

$g(x) = \dfrac{1}{1-x}$

Observe $L = \lim\limits_{x\to 1^-} \dfrac{\frac{-\sin x}{\ln x}}{\frac{1}{1-x}} = \lim\limits_{x\to 1^-} \dfrac{-(1-x)}{\ln x} \underbrace{\sin(x)}_{\text{some finite number}}$

$\lim\limits_{x\to 1^-} \dfrac{+(1-x)}{\ln x} \underset{LR}{=} \lim\limits_{x\to 1^-} \dfrac{1}{1/x} = 1$

Also, $\int_{1/2}^{1} \frac{1}{1-x}\,dx = \lim\limits_{t\to 1^-} \int_{1/2}^{t} \frac{1}{1-x}\,dx \overset{\substack{u=1-x \\ du=-dx}}{=} \lim\limits_{t\to 0^+} \int_t^{1/2} \frac{1}{u}\,du = \int_0^{1/2} \frac{1}{u}\,du$ which is divers. by p-test $p=1$

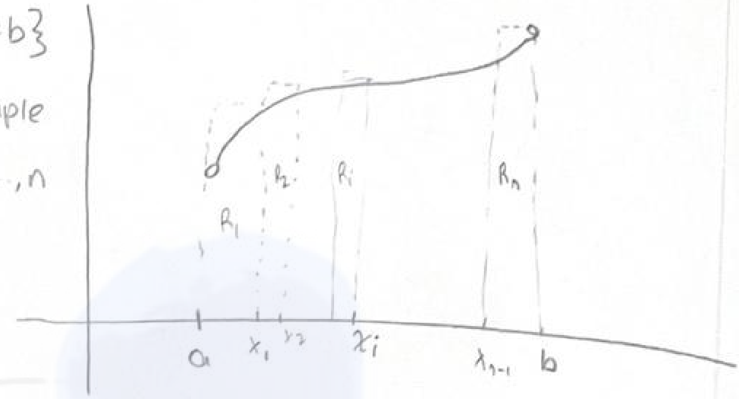
$\therefore$ I is divergent by LCT

Aplications of Integration

Area Between Curves

Goal: Given $y = f(x)$, $y = g(x)$ on $[a,b]$, we want to define and compute the area between the graphs of $f(x)$ and $g(x)$.

Let $P = \{x_0 = a, x_1, \ldots, x_i, \ldots, x_{n-1}, x_n = b\}$ be a partition of $[a,b]$. Choose a sample point $x_i^* \in [x_{i-1}, x_i]$ for each $i = 1, \ldots, n$



If A exists, then

$$A \approx \sum_{i=1}^{n} \text{Area}(R_i)$$

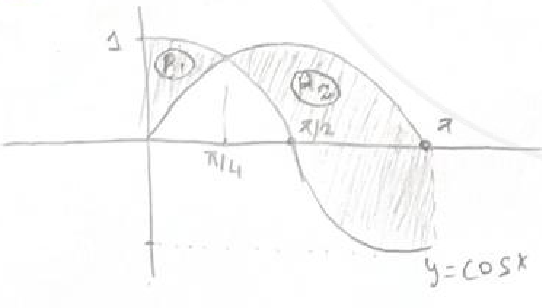
$$= \underbrace{\sum_{i=1}^{n} |f(x_i^*) - g(x_i^*)| \Delta x}_{\text{Rienmann sum of } |f(x) - g(x)|}$$

So, these suggest

Def: $A \overset{\text{def}^n}{=} \lim_{n\to\infty} \sum_{i=1}^{n} |f(x_i^*) - g(x_i^*)| \Delta x_i = \int_a^b |f(x) - g(x)| dx$ provided the limit exists

example Find the area A between the graphs of the given functions.

1. $f(x) = \cos x$ $g(x) = \sin x$ over $[0, \pi]$



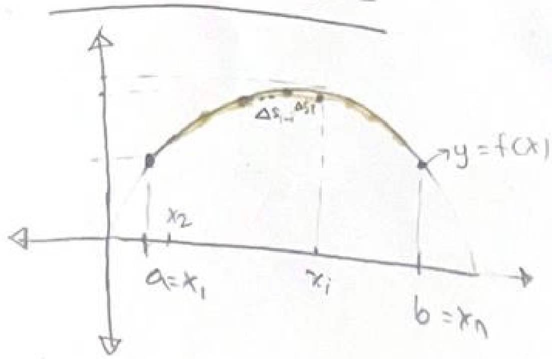
$\cos x = \sin x \Rightarrow x = \frac{\pi}{4}$

$A = A_1 + A_2$

$$\int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx = \int_0^{\pi} |\sin(x) - \cos(x)| dx$$

$$= 2\sqrt{2}$$

Proof of case-I



(use sec. lines)

Δx is always positive by definition.

$\{x_0=a, x_1, x_2, \dots, x_{i-1}, x_i, \dots, x_{n-1}, x_n=b\}$ is a partition of $[a,b]$

for each $i=1,\dots,n$, $(\Delta s_i)^2 = (\Delta x_i)^2 + (\Delta y_i)^2$

$$\Rightarrow \Delta s_i = \sqrt{(\Delta x_i)^2\left(1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2\right.}$$

$$= \Delta x_i \sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)}$$

By Mean value Theorem, $\exists\, x_i^* \in (x_{i-1}, x_i)$ s.t

$$f'(x_i^*) = \frac{\Delta y_i}{\Delta x_i}$$

So, if L exists, then

$$L \approx \sum_{i=1}^{n} \Delta s_i = \underbrace{\sum_{i=1}^{n} \sqrt{1+(f'(x_i^*))^2}\,\Delta x_i}_{\text{a Riemann Sum of } \sqrt{1+(f'(x_i^*))^2}}$$

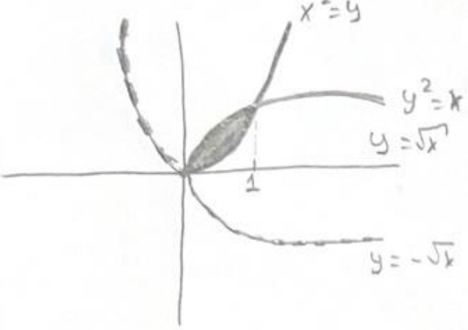
$$L \overset{def}{=} \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

provided the limit of Riemann Sum exists.

② $y = x^2$ & $y^2 = x$

Soln

1^{st} step is drawing a picture.



$$A = \int_0^1 (\sqrt{x} - x^2)\,dx = \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3} \right]\Bigg|_0^1 = 1/3$$

③ $y = \sin\left(\frac{\pi x}{2}\right)$ & $y = x$

$\sin\left(\frac{\pi x}{2}\right) = x \qquad x \in \{-1, 0, 1\}$

By symmetry, $A = 2 \int_0^1 \left(\sin\left(\frac{\pi x}{2}\right) - x\right) dx$

$$= \frac{4}{\pi} - 1$$

(exercise: Find the area of the region enclosed by the curves $y = 8 - x^2$, $y = x^2$, $x = -3$ & $x = 3$. (Answer: $\frac{92}{3}$)

Arc Lenght

Goal: Given a curve $\mathcal{C}$ in the x-y plane we want to define and compute the lenght L of $\mathcal{C}$ using a definite integral.

We have two basic cases

General curves can be decomposed into parts so that each part falls into one of these two basic cases.

Case - I

$\mathcal{C}: y = f(x),\ a \leq x \leq b$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$ where f is cont. diff'able.

Case - II

$\mathcal{C}: x = g(y),\quad c \leq g(y) \leq d$

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\,dy$$ where g is cont. diff'able