Triple Integrals

Graphs that should be known beforehand:

① Planes

$ax+by+cz+d=0$

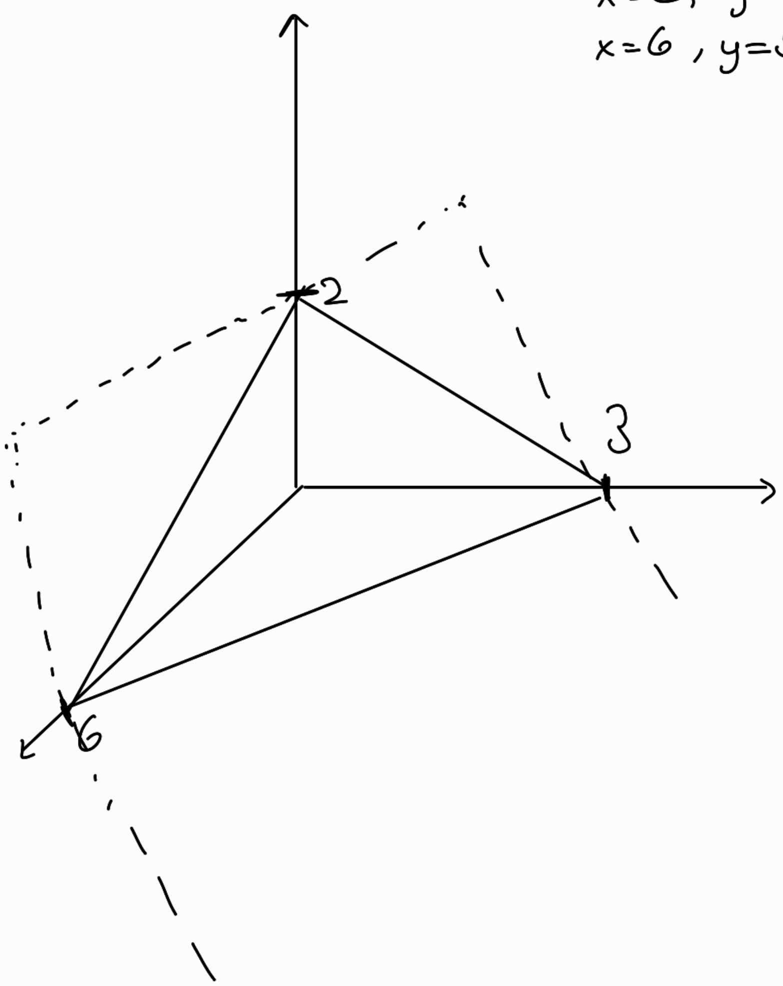
→ eqn.'s of the form:

$x=a$, $ax+by=k$
$y=b$, $ax+cz=l$
$z=c$, $by+cz=m$

also counts.

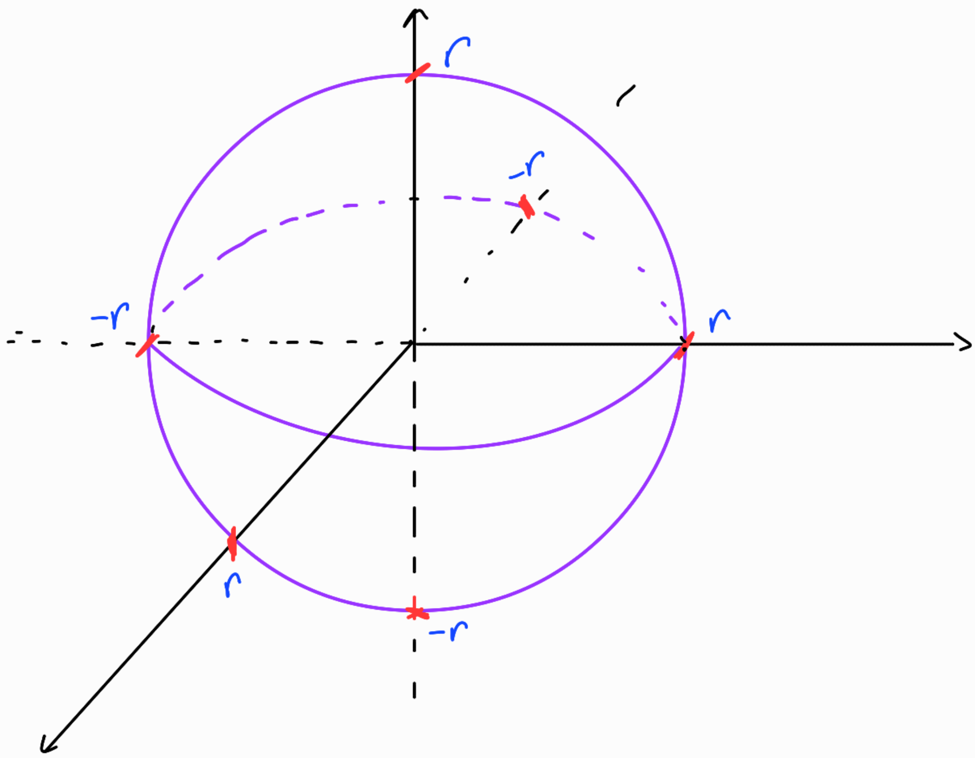
e.g. $x+2y+3z=6$ → $x=0,\ y=0,\ z=2$ — $(6,0,0)$
$x=0,\ y=3,\ z=0$ — $(0,3,0)$
$x=6,\ y=0,\ z=0$ — $(0,0,2)$

points where it cuts the axes.



② Spheres

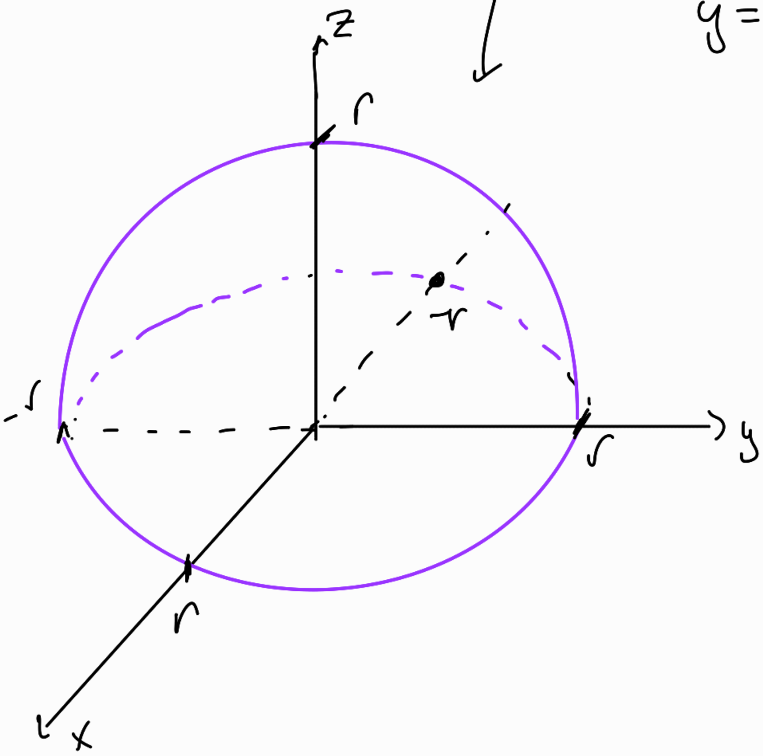
$x^2+y^2+z^2=r^2$



③ Hemispheres

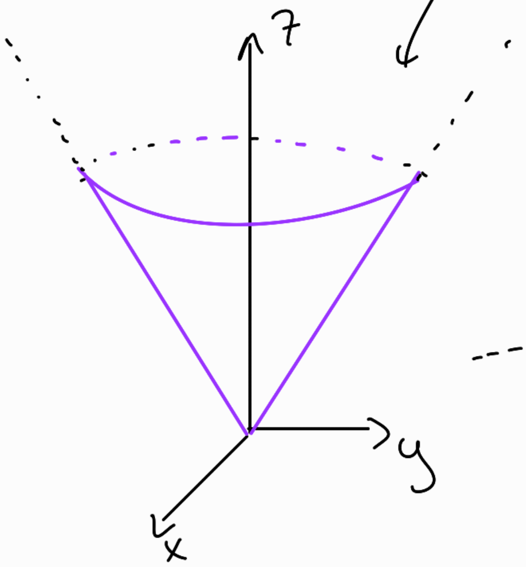
$z=\sqrt{r^2-x^2-y^2}$ $x=\sqrt{r^2-y^2-z^2}$

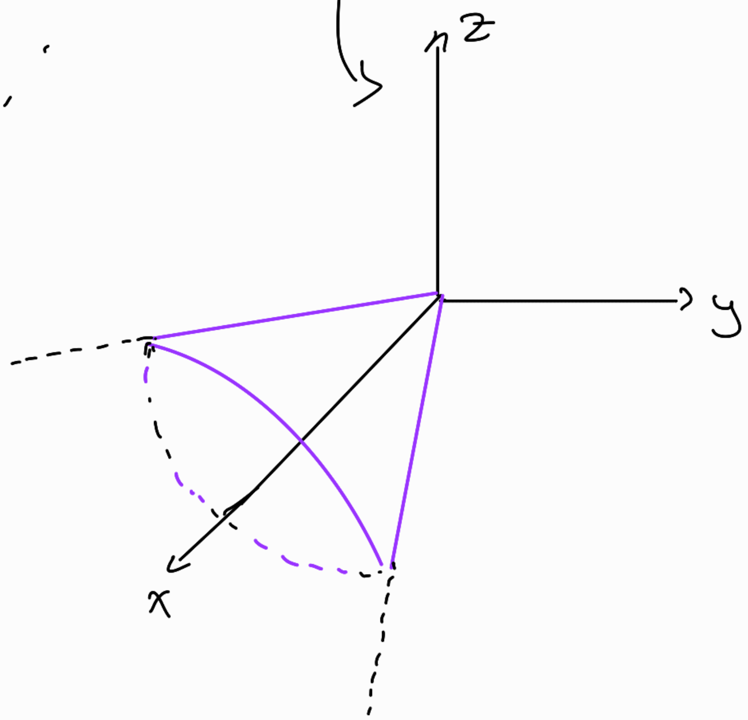
$y=\sqrt{r^2-z^2-x^2}$



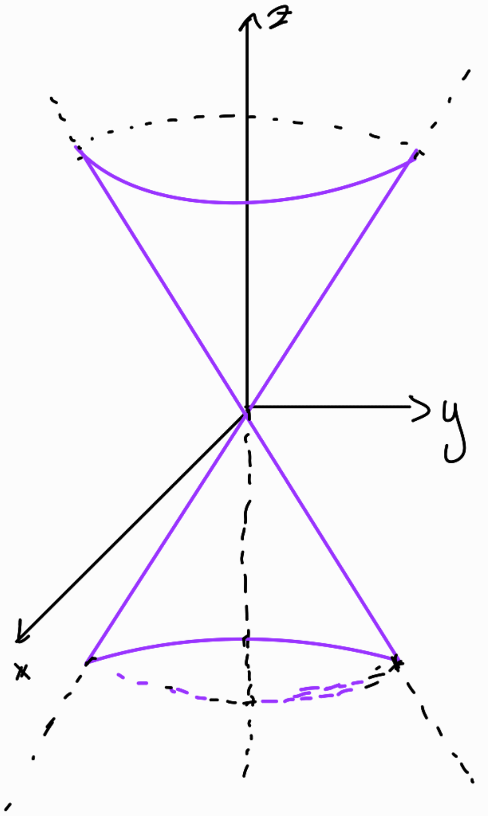
④ Cones

$z = \sqrt{x^2+y^2}$ $\quad x = \sqrt{z^2+y^2}$ $\quad y = \sqrt{z^2+x^2}$





⑤ Double Cones



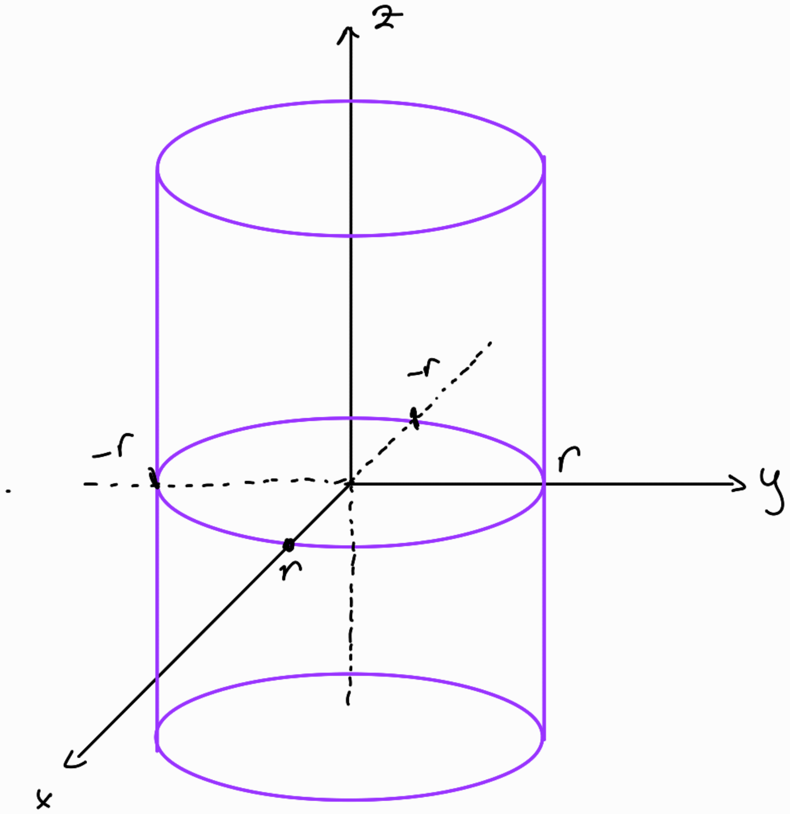
$z^2 = \sqrt{x^2+y^2}$

$x^2 = \sqrt{y^2+z^2}$

$y^2 = \sqrt{x^2+z^2}$

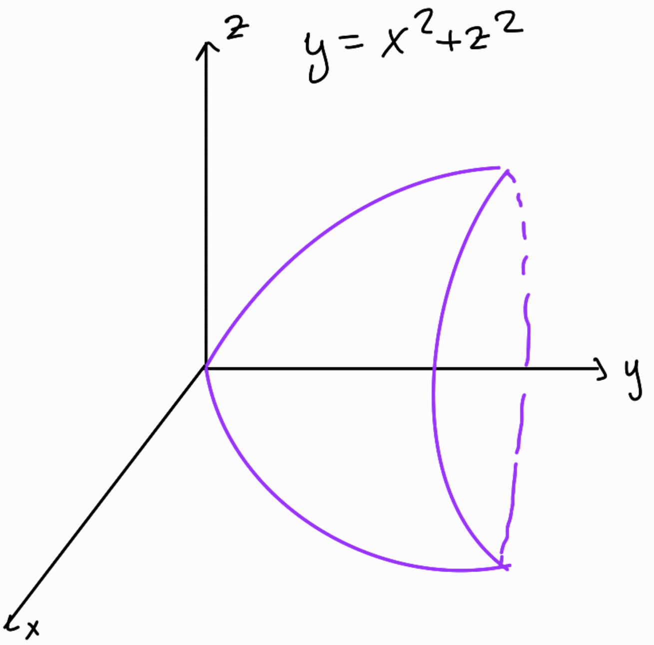
⑥ Cylinders

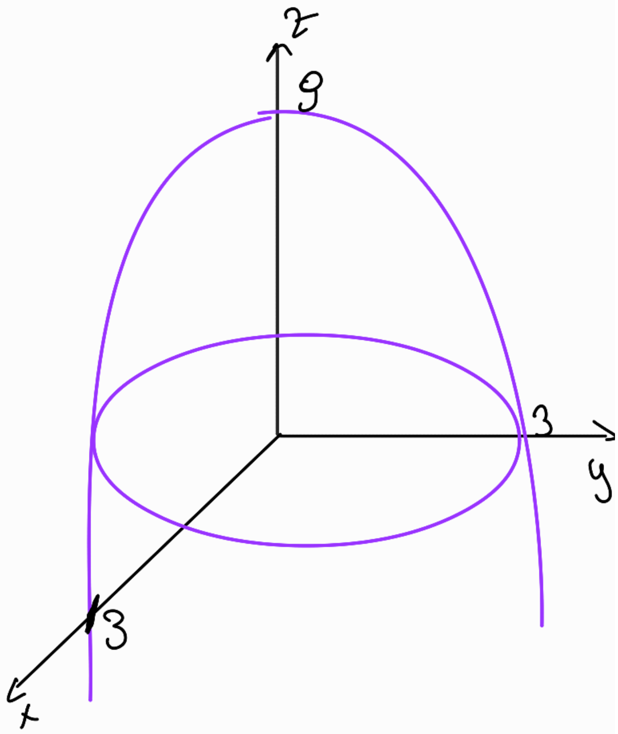
$x^2+y^2=r^2$, $x^2+z^2=r^2$, $y^2+z^2=r^2$



⑦ Paraboloids

$x=y^2+z^2$ $\qquad$ $z=9-x^2-y^2$





Cylindrical Coordinates

$(x, y, z) \longrightarrow (r, \theta, z)$

cartesian coord. — cylindrical coord

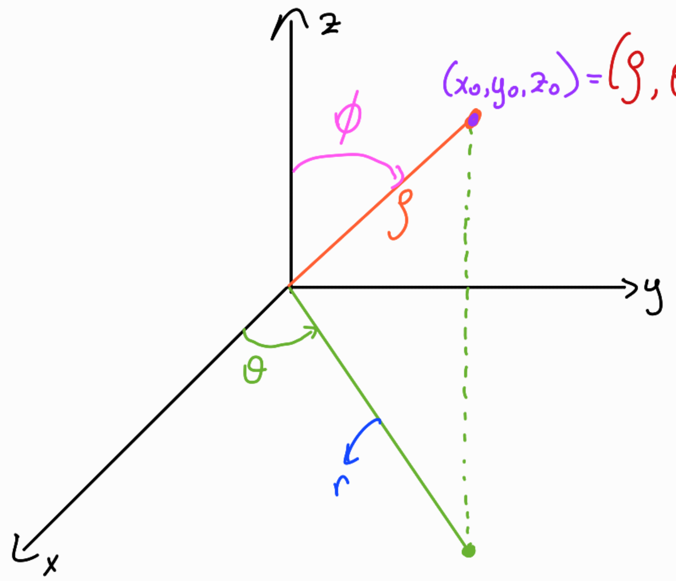
- $x = r\cos\theta$ $\quad z = z \quad x^2 + y^2 = r^2 \quad dz\,dy\,dx = r\,dz\,dr\,d\theta$

  $y = r\sin\theta$

$$\boxed{\iiint_D f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\int_{z_1}^{z_2} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta}$$

Spherical Coordinates

$(x,y,z) \longrightarrow (\rho, \theta, \phi)$



$$\rho = \sqrt{x^2+y^2+z^2}$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \pi$$

$x_0 = r \cdot \cos\theta$ and $r = \rho \cdot \sin\phi$

$$\boxed{x_0 = \rho \cdot \cos\phi \cdot \sin\theta}$$

why?

$y_0 = r \cdot \sin\theta$

$$\boxed{y_0 = \rho \cdot \sin\phi \cdot \sin\theta}$$

$$\boxed{z_0 = \rho \cos\phi}$$

$$\boxed{x^2+y^2+z^2 = \rho^2}$$

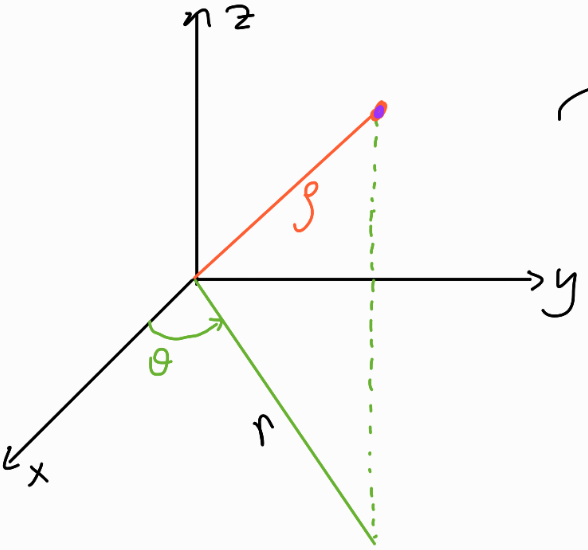
$(\rho, \phi, \theta) \xrightarrow{\mathcal{U}} (x,y,z)$

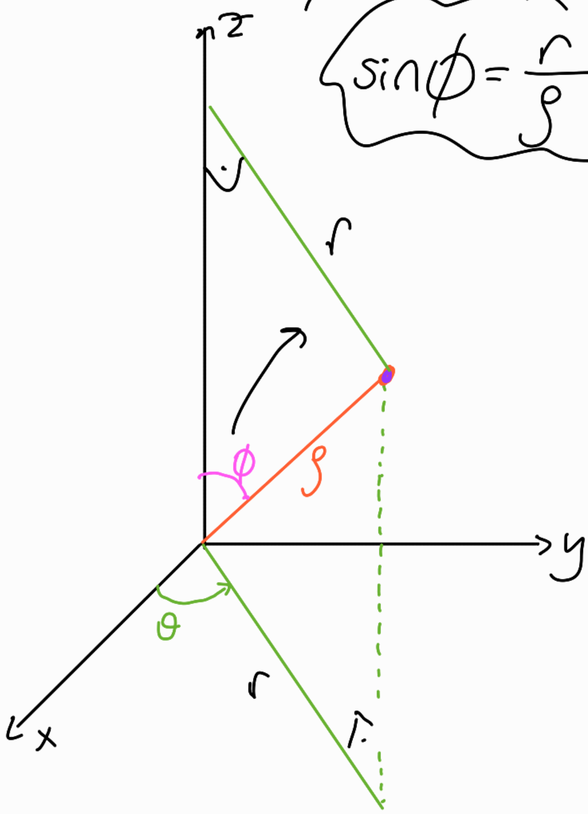
$\mathcal{U}$ is 1-1 : $\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$

$0 \leq \rho < \infty$
$0 \leq \phi \leq \pi$
$0 \leq \theta \leq 2\pi$

$$\iiint_E f(x,y,z)\, dV = \int_{\phi} \int_{\theta} \int_{\rho} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

→ Why is $r = \rho \sin\phi$?





$$\sin\phi = \frac{r}{\rho}$$

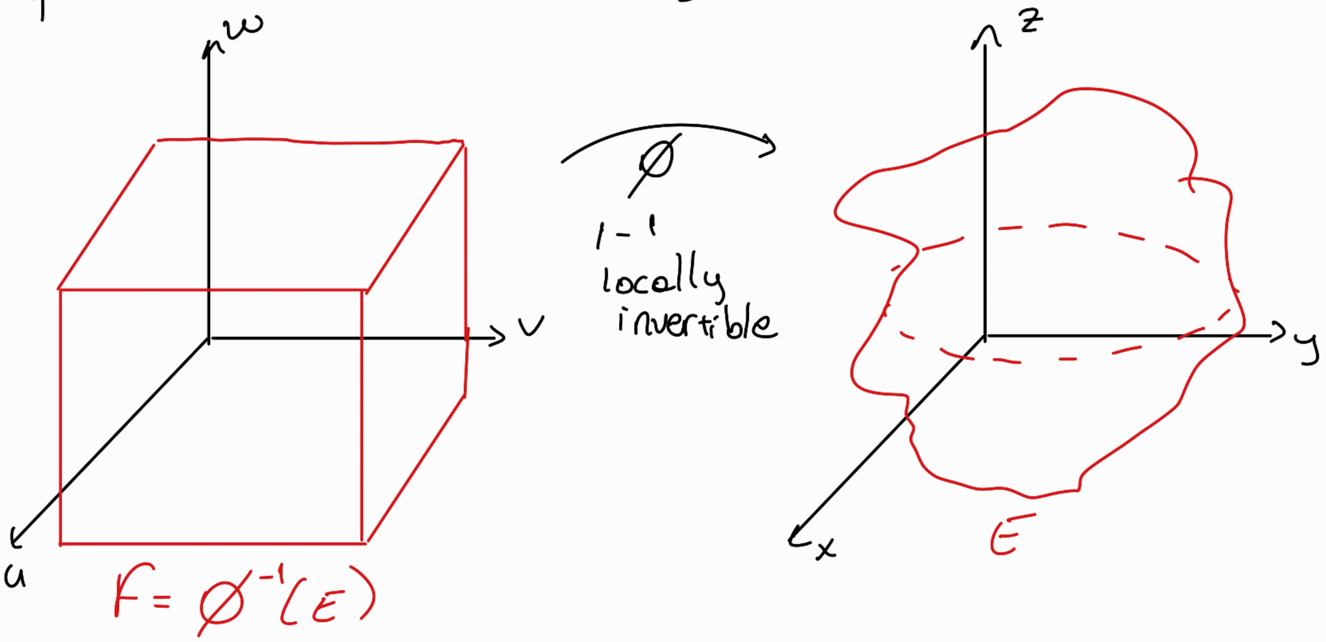
Since $\sin\phi = \frac{r}{\rho}$

$r = \rho . \sin\phi$

similar for z.

Change of Coordinates in Triple Integrals

$$\iiint_F g(u,v,w)\,du\,dv\,dw \overset{?}{\longleftrightarrow} \iiint_E f(x,y,z)\,dx\,dy\,dz$$



ϕ diff'able

$|d\phi| \neq 0$ in this region F

$$\phi = \begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases} \qquad d\phi = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

$$dV = dx\,dy\,dz = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$