Double Integrals

$$\underset{\text{Riemann sum}}{S}(f, N, \{P_{ij}\}) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}) \overbrace{A(R_{ij})}^{\text{Area of } R_{ij}}$$

- if $f(x,y)$ is integrable over R, we write

$$\iint_R f = \lim_{d(N)\to 0} S(f, N, \{P_{ij}\})$$

* If f is continuous on R, then $\iint_R f$ exists.

Properties Let f & g be integrable over D & L, M be constants.

- $\iint_D (f(x,y)\,dA) = 0$ if D has zero area (line)
- $\iint_D 1\,dA = \text{Area}(D)$
- If $f(x,y) \geq 0$ on D, then $\iint_D f(x,y)\,dA = V \geq 0$
- If $f(x,y) \leq 0$ on D, then $\iint_D f(x,y)\,dA = -V \leq 0$

- $\iint_D (L f(x,y) + M g(x,y))\,dA = L \iint_D f(x,y)\,dA + M \iint_D g(x,y)\,dA$

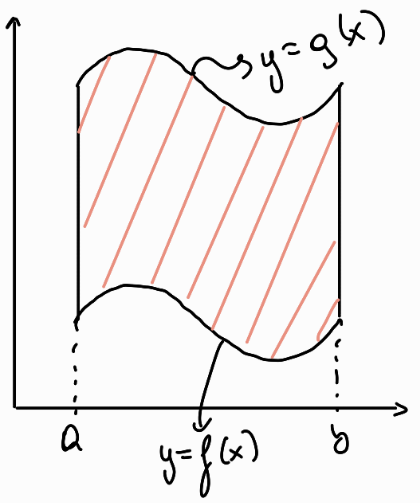
- If $f(x,y) \le g(x,y)$ on D, then $\iint_D f(x,y)\,dA \le \iint_D g(x,y)\,dA$.

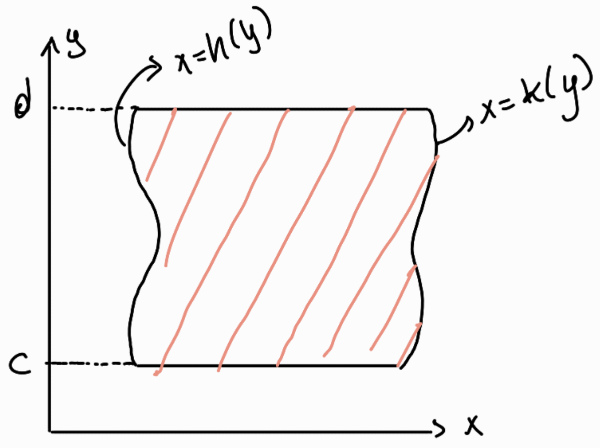
- $\left| \iint_D f(x,y)\,dA \right| \le \iint_D |f(x,y)|\,dA$

- $\iint_D f(x,y)\,dA = \sum_{i=1}^{k} \iint_{D_i} f(x,y)\,dA$

$D = D_1 \cup D_2 \cup \ldots \cup D_k$

Domain of Double Integrals;





y simple domain

For a cont. func. on y simple domain:

$$\iint_D f(x,y)\,dA = \int_a^b \int_{f(x)}^{g(x)} f(x,y)\,dx\,dy$$

x simple domain

For a cont. func. on x simple domain:

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h(y)}^{k(y)} f(x,y)\,dx\,dy$$

• In special cases where the integrand $f(x,y)$ can be factored as a product of a func. of x only & a func. of y only. The double integral over rectangles can be written in a simpler form:

$f(x,y) = g(x).h(y)$ $D = R = [a,b] \times [c,d]$ then

$$\iint_R f(x,y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

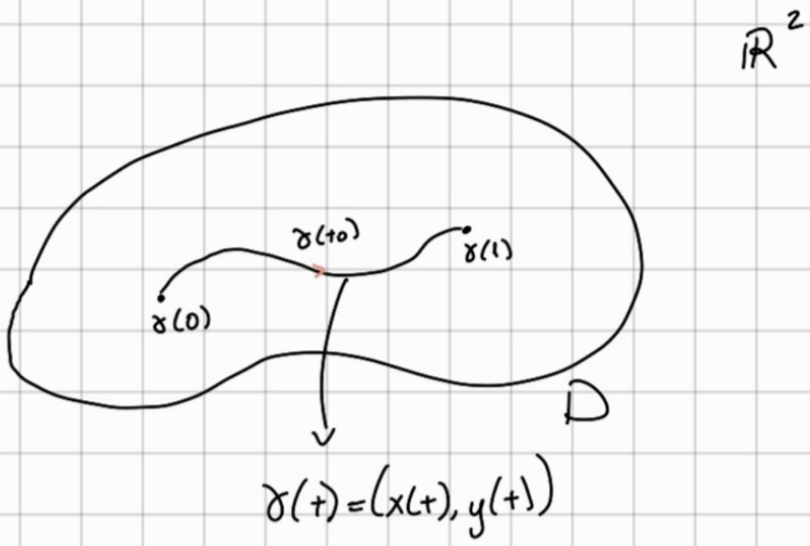
Thm (M.V.T.D.I)

Let $f(x,y)$ be a cont. func. on a domain D with finite area (closed, bounded and connected subset of $\mathbb{R}^2$), then $\exists\,(x_0,y_0) \in D$ such that

$$\boxed{\iint_D f(x,y)\,dA = \text{Area}(D)\cdot f(x_0,y_0)}$$

Defn: A subset D of $\mathbb{R}^2$ is called "connected" if you can join any 2 pt's in D by a "path" in D.

Defn: A path, γ, in a subset D of $\mathbb{R}^2$ is just the image of a cont. func. $\gamma : [0,1] \to D \subseteq \mathbb{R}^2$



Change of Variables in Double Integrals

$$\iint_R f(x,y)\,dx\,dy \qquad (x,y) \to (u,v)$$

$$\Big\Downarrow \quad \begin{aligned} x &= g(u,v) \\ y &= h(u,v) \end{aligned}$$

$$\hookrightarrow \iint_S f(g(u,v), h(u,v))\,|J|\,du\,dv$$

$\hookrightarrow$ Jacobian det.

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

Special Case : Polar Coordinates

Recall Circle eqn. $\rightarrow (x-a)^2+(y-b)^2=r^2$

a circle with radius r, centre at (a,b)

When to change? (to polar coord.s)

Domain like $x^2+y^2=r^2$

or with $\left.\begin{aligned} y &= \sqrt{r^2-x^2} \\ x &= \sqrt{r^2-y^2} \end{aligned}\right\}$ etc.

continued $\Longrightarrow$

How to change?

1) $x = r\cos\theta$

$y = r\sin\theta$

2) $x^2+y^2=r^2$

3) $dx\,dy = r\,dr\,d\theta$

Why?

bcs

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}$$

$$= r$$

$$\boxed{\iint_D f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta}$$

Integrals with Ellipse Domains

Recall: Ellips formula: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

ellipse which passes through $(a,0),(-a,0)$ and $(0,b),(0,-b)$.

Idea: Transform the ellipse into a circle via change of coordinates. Then use polar coord. transformation.

In general, for $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, say $x=au$, $y=bv$

so the eqn. becomes $\frac{\cancel{a^2}u^2}{\cancel{a^2}}+\frac{\cancel{b}v^2}{\cancel{b^2}}=1$, so we have

$u^2+v^2=1$, $\therefore$ we have the unit circle in u-v coord. system.

to sum up:

$$\iint_D f(x,y)\,dA=\iint_S f(au,bv)|J|\,du\,dv=\iint_R f(a.r\cos\theta,b.\sin\theta)\,r\,dr\,d\theta.$$

$D=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Do NOT forget the Jacobian.