Arc Length

Let C be bounded, continuous curve

$$r = r(t) \quad , a \le t \le b,$$

subdivide the closed interval $[a,b]$ into n subintervals by points

$$a = t_0 < t_1 < t_2 \cdots < t_{n-1} < t_n = b$$

The points $r_i = r(t_i)$, $(0 \le i \le n)$, subdivide C into n arcs. If we use the chord length $|r_i - r_{i-1}|$ as an approximation to the arc length between r_{i-1} and r_i, then the sum

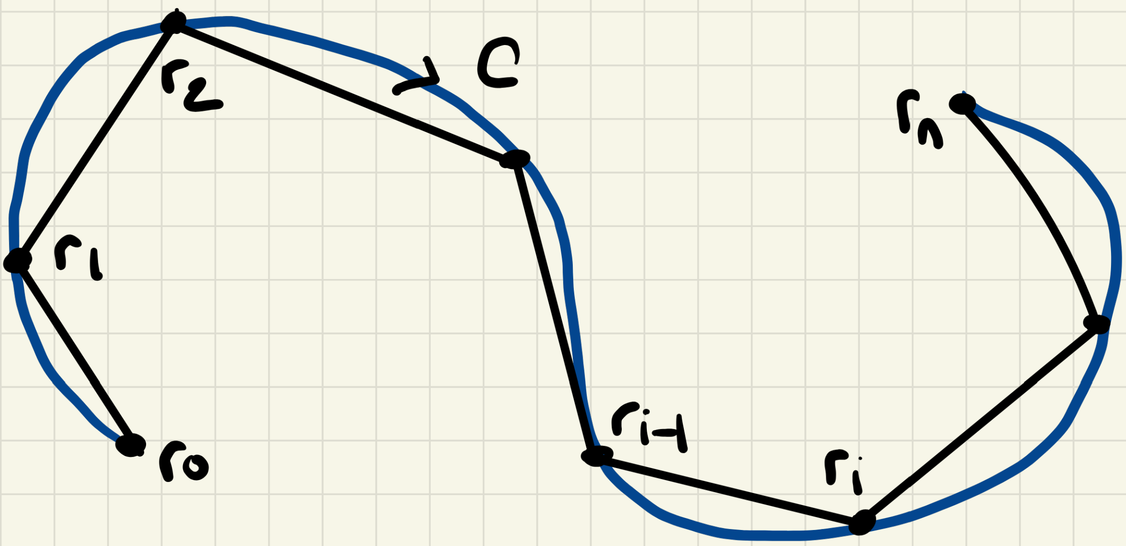
$$\boxed{s_n = \sum_{i=1}^{n} |r_i - r_{i-1}|}$$

approximates the length of C by the length of a polygonal line.

→ C is rectifiable if there exists a constant K such that $s_n \le K$ for every n and every choice of the points t_i. We will call this smallest K the length of C and denote it by s.

Let $\Delta t_i = t_i - t_{i-1}$ and $\Delta r_i = r_i - r_{i-1}$. Then s_n can be written in the form

$$\boxed{s_n = \sum_{i=1}^{n} \left| \frac{\Delta r_i}{\Delta t_i} \right| \Delta t_i}$$



If $r(t)$ has a continuous derivative $v(t)$ then

$$\boxed{s = \lim_{\substack{n\to\infty \\ \max \Delta t_i \to 0}} s_n = \int_a^b \left| \frac{dr}{dt} \right| dt = \int_a^b |v(t)|\, dt = \int_a^b v(t)\, dt}$$

If $s(t)$ denotes the arc length of that part of C corresponding to parameter values in $[a,t]$, then

$$\frac{ds}{dt} = \frac{d}{dt}\int_a^t v(\tau)\,d\tau = v(t)$$

so that the arc length element for C is given by

$$\boxed{ds = v(t)\,dt = \left|\frac{d}{dt}\,r(t)\right| dt}$$

The length of C is the integral of these arc length elements; we write

$$\int_C ds = \text{length of } C = \int_a^b v(t)\,dt$$

Several familiar formulas for arc length follow from the above formula by using specific parametrizations of curves.

- $r = xi + f(x)j$, so $v = i + f'(x)j$ and

$$\boxed{ds = \sqrt{1+(f'(x))^2}\,dx}$$

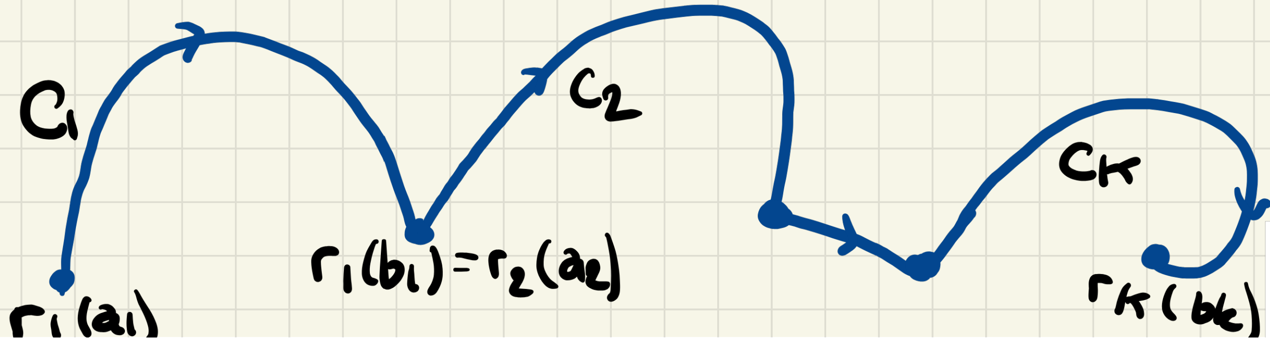
- (polar curves) $r = g(\theta)$ can be calculated from the parametrization

$$r(\theta) = g(\theta)\cos\theta\, i + g(\theta)\sin\theta\, j$$

$$\boxed{ds = \sqrt{(g(\theta))^2 + (g'(\theta))^2}\,d\theta}$$

Piecewise Smooth Curves

A parametric curve C given by $r = r(t)$ can fail to be smooth at points where $dr/dt = 0$. If there are finitely many such points, we will say that curve is piecewise smooth.



In this case we express C as the sum of the individual arcs:

$$C = C_1 + C_2 + \cdots + C_k$$

Each arc C_i can have its own parametrization

$$r = r_i(t), \quad (a_i \le t \le b_i),$$

where $v_i = dr_i/dt \neq 0$ for $a_i < t < b_i$.

The fact that C_{i+1} must begin at the point where C_i ends requires the conditions

$$r_{i+1}(a_{i+1}) = r_i(b_i) \quad \text{for } 1 \le i \le k-1$$

If also $r_k(b_k) = r_1(a_1)$, then C is a closed piecewise smooth curve.

The length of a piecewise smooth curve $C = C_1 + C_2 + \cdots + C_k$ is the sum of the lengths of its component arcs:

$$\boxed{\text{length of } C = \sum_{i=1}^{k} \int_{a_i}^{b_i} \left| \frac{dr_i}{dt} \right| dt}$$

The Arc-Length Parametrization

The position vector of an arbitrary point P on the curve can be specified as a function of the arc length s along the curve from the initial point P_0 to P,

$$\boxed{r = r(s)}$$

This eqn is called an arc-length parametrization or intrinsic parametrization of the curve.

Since $ds = \sigma(t)\,dt$ for any parametrization $r = r(t)$, for the arc length parametrztn we have $ds = \sigma(s)\,ds$. Thus $\sigma(s) = 1$, identically; a curve parametrized in terms of arc length is traced at unit speed.

Suppose that a curve is specified in terms of an arbitrary parameter t. If the arc length over a parameter interval $[t_0, t]$,

$$\boxed{s = s(t) = \int_0^t \left| \frac{d}{d\tau} r(\tau) \right| d\tau}$$

can be evaluated.