ORTA DOĞU TEKNİK ÜNİVERSİTESİ

MATEMATİK TOPLULUĞU

MATH 114- MIDTERM 1 ÖZETİ

ODTÜ Matematik Topluluğu 2025

Integration by Parts

$$\int f(x)\cdot g'(x)\,dx = f(x)g(x) - \int f'(x)\,g(x)\,dx$$

$\rightarrow$ Letting $\begin{matrix} u = f(x) \\ v = g(x) \end{matrix} \Rightarrow \begin{matrix} du = f'(x)\,dx \\ dv = g'(x)\,dv \end{matrix}$

$$\rightarrow \int u\,dv = uv - \int v\,du \qquad \text{OR} \qquad \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v.du$$

(for definite integrals)

Integrals of Rational Functions

Given $I = \int \frac{P(x)}{Q(x)}\,dx$ where $P(x)$ & $Q(x)$ are polynomials:

If $\deg(P) \geq \deg(Q)$, use long division

$$\begin{array}{r|l} P(x) & Q(x) \\ \cline{2-2} & S(x) \\ \cline{1-1} R(x) & \end{array} \quad \dashrightarrow \quad \int \frac{P(x)}{Q(x)}\,dx = \int S(x)\,dx + \int \frac{R(x)}{Q(x)}\,dx$$

If $\deg(P) < \deg(Q)$, then use partial fraction method:

	Factor of $Q(x)$	Corresponding partial fraction contribution
linear factor	$(ax+b)^r$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_r}{(ax+b)^r}$
irreducible quadratic factor ($\dashrightarrow \Delta < 0$)	$(ax^2+bx+c)^r$	$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$

Trigonometric Integrals

Strategy for: $\int \sin^m x \cos^n x \, dx$ $(m, n \in \mathbb{Z}^+)$

i) <u>n is odd:</u> Save one $\cos x$, use $\cos^2 x = 1 - \sin^2 x$, and let $u = \sin x$.

ii) <u>m is odd:</u> Save one $\sin x$, use $\sin^2 x = 1 - \cos^2 x$, and let $u = \cos x$.

iii) <u>m,n are even:</u> use half-angle identities such as:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x), \quad \sin x \cos x = \frac{1}{2}\sin 2x.$$

Strategy for: $\int \tan^m x \sec^n x \, dx$ $(m, n \in \mathbb{Z}^+)$

i) <u>n is even:</u> Save a $\sec^2 x$ and use $\sec^2 x = 1 - \tan^2 x$, let $u = \tan x$

ii) <u>m is odd:</u> Save a $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$, let $u = \sec x$.

Strategy for

$$\left\{ \begin{array}{l} \int \sin(mx)\cos(nx)\,dx \\ \int \sin(mx).\sin(nx)\,dx \\ \int \cos(mx).\cos(nx)\,dx \end{array} \right. \Rightarrow$$

<u>Use the following:</u>

- $\sin\alpha.\cos\beta = \frac{1}{2}\left[\sin(\alpha - \beta) + \sin(\alpha + \beta)\right]$
- $\sin\alpha.\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$
- $\cos\alpha.\cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$

Reduction Formulas: (for $n \in \mathbb{Z}^+$)

1- $\int \sin^n x\,dx = \frac{-1}{n} \sin^{n-1} x \,.\, \cos x + \frac{n-1}{n} \int \sin^{n-2} x\,dx$

2- $\int \tan^n x\,dx = \frac{1}{n-1} \tan^{n-1} x\,dx - \int \tan^{n-2} x\,dx$

3- $\int \sec^n x\,dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n+2}{n+1} \int \sec^{n-2} x\,dx.$

Inverse Trigonometric Substitutions

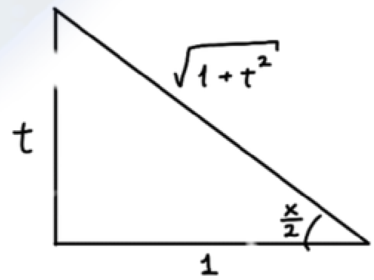
expression appeared in the integrand		corresponding substitution
$a^2 - x^2$	$\longrightarrow$	$x = a\sin\theta$
$x^2 - a^2$	$\longrightarrow$	$x = a\sec\theta$
$a^2 + x^2$	$\longrightarrow$	$x = a\tan\theta$

$\tan\left(\frac{x}{2}\right)$ substitution

$\rightarrow$ Useful with rational expressions involving trigonometric functions.

Let $t = \tan\left(\frac{x}{2}\right) \Rightarrow dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$

where $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$ $\quad \hookrightarrow dt = \frac{1}{2} \cdot (1+t^2)dx \Rightarrow \frac{2dt}{1+t^2} = dx$



$\rightarrow$ We can find values of $\sin\left(\frac{x}{2}\right)$, $\sin x$, $\cos\left(\frac{x}{2}\right)$, $\cos x$, etc. using the triangle

e.g. $\sin x = 2 \cdot \sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2}\right)$

$$= \frac{2 \cdot t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$$

P-test for improper integrals

Suppose $a>0$, then

i) $$\int_a^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{Convergent if } p>1 \\ \text{divergent if } p\leq 1 \end{cases}$$

ii) $$\int_0^a \frac{1}{x^p}\,dx = \begin{cases} \text{Convergent if } p<1 \\ \text{divergent if } p\geq 1 \end{cases}$$

Comparison test (CT)

Suppose $0\leq f(x)\leq g(x)$ $\forall x\in[a,\infty)$

i) $\int_a^\infty g(x)\,dx$ is convergent $\Rightarrow \int_a^\infty f(x)\,dx$ is convergent.

(Type-2 case is similar)

ii) $\int_a^\infty f(x)\,dx$ is divergent $\Rightarrow \int_a^\infty g(x)\,dx$ is divergent.

Absolute convergent theorem (ACT)

$\int_a^\infty |f(x)|\,dx$ is convergent $\Rightarrow \int_a^\infty f(x)\,dx$ is convergent.

(For type-2)

$\int_a^b |f(x)|\,dx$ is convergent $\Rightarrow \int_a^b f(x)\,dx$ is convergent.

(if $f(x)$ is continuous on $[a,b)$ and has infinite discontinuity at b.)

Area between $f(x)$ and $g(x)$ from a to b:

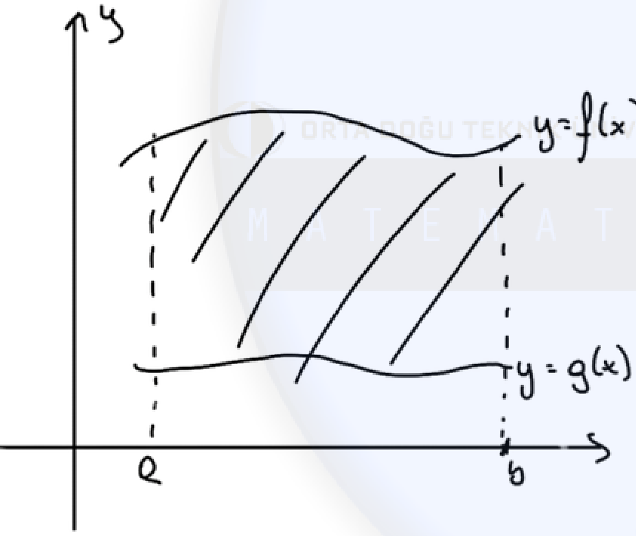
$$\overset{\text{if exists}}{=} \int_a^b |f(x)-g(x)|\,dx$$

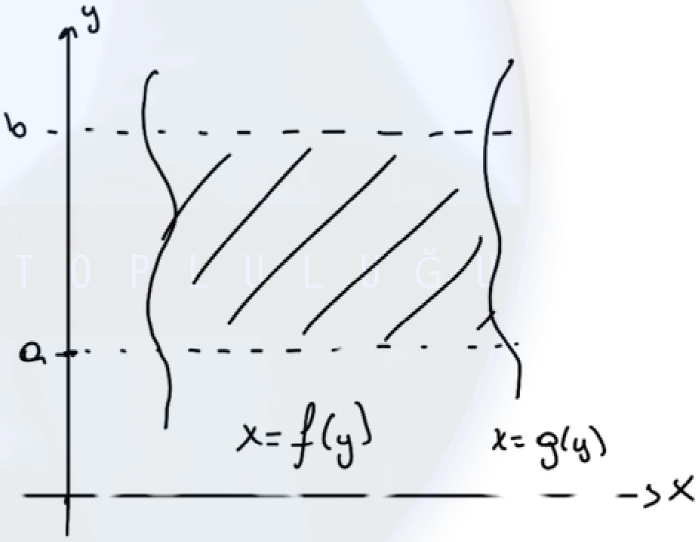
Disk (or special slicing) method

If S is the solid obtained by rotating a plane region D, such that

$D = \{(x,y) \in \mathbb{R}^2 \mid g(x) \leq y \leq f(x),\ a \leq x \leq b\}$ about the x-axis, then

$$Vol(S) = \int_a^b \pi \left[(f(x))^2 - (g(x))^2 \right] dx$$





$$Vol(S) = \int_a^b \pi \left[(f(x))^2 - (g(x))^2 \right] dx$$

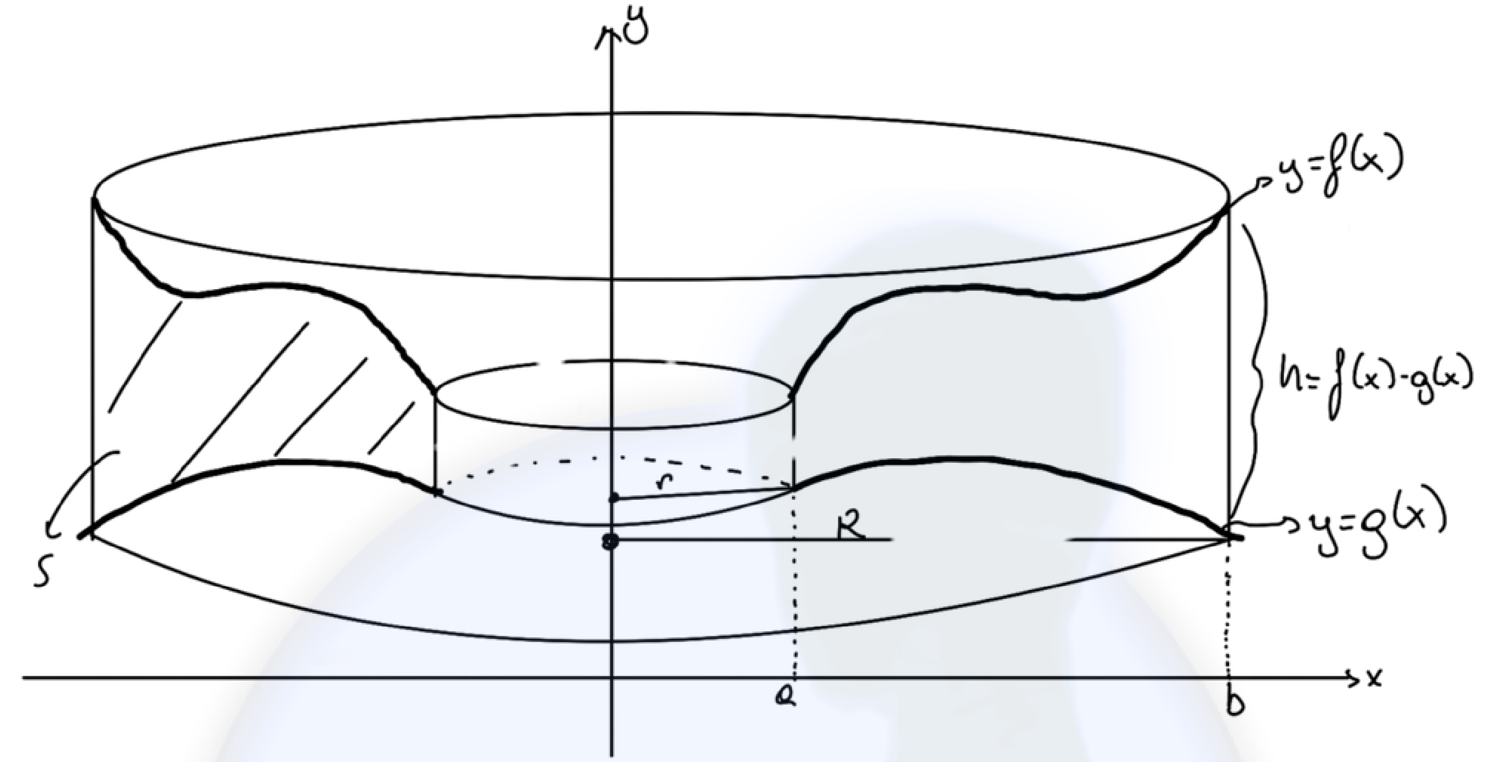
(around the x-axis)

$$Vol(S) = \pi \int_a^b \left[(f(y))^2 - (g(y))^2 \right] dy$$

(around the y-axis).

Remark: Rectangles are perpendicular to the rotation axis.

Cylindrical shell method



r: shell radius
h = shell height

$$Vol(S) = 2\pi \int_a^b x\,(f(x)-g(x))\,dx$$

$$\Delta V = 2\pi \cdot \overset{x}{r} \cdot \underset{f(x)-g(x)}{h} \cdot \Delta x$$

Remark: Rectangles // rotation axis
↳ x-axis, in this case.

Arc Length Given a curve C in the xy-plane, we use definite integral to calculate the length L of C

Case-1

$C: y=f(x),\ a \leq x \leq b$

$$\Rightarrow L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

Case-2

$C: x=g(y),\ c \leq y \leq d$

$$\Rightarrow L = \int_c^d \sqrt{1+(g'(y))^2}\,dy$$

(provided f & g are continuously differentiable)

Areas of Surface of Revolutions

Let A_S denote the area of the surface S obtained by rotating the given curve G about the given axis. Then, A_S can be computed via the table below:

the curve G we rotate	Case-1 (rotation about the x-axis)	Case-2 (rotation about the y-axis)
$y=f(x)$ $a \leq x \leq b$	$A_S=\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$	$A_S=\int_a^b 2\pi x\sqrt{1+(f'(x))^2}\,dx$
$x=g(y)$ $c \leq y \leq d$	$A_S=\int_c^d 2\pi y\sqrt{1+(g'(y))^2}\,dy$	$A_S=\int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy$

!? Provided f & g are continuously differentiable.