15.1 – 15.3

Vector Fields

Line Integrals

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MATEMATİK TOPLULUĞU

15.1

Vector and Scalar Fields

A function whose domain and range are subsets of Euclidean 3-space, $\mathbb{R}^3$, is called a vector field.

Thus, a vector field F associates a vector $F(x,y,z)$ with each point (x,y,z) in its domain. The three components of F are scalar-valued (real-valued) functions $F_1(x,y,z)$, $F_2(x,y,z)$ and $F_3(x,y,z)$ and $F(x,y,z)$ can be expressed in terms of the standard basis in $\mathbb{R}^3$ as

$$F(x,y,z) = F_1(x,y,z)\,i + F_2(x,y,z)\,j + F_3(x,y,z)\,k$$

If $F_3(x,y,z) = 0$ and F_1 an F_2 are independent of z, then F reduces to

$$F(x,y,z) = F_1(x,y,z)\,i + F_2(x,y,z)\,j$$

and so is called a plane vector field or vector field in the xy-plane.

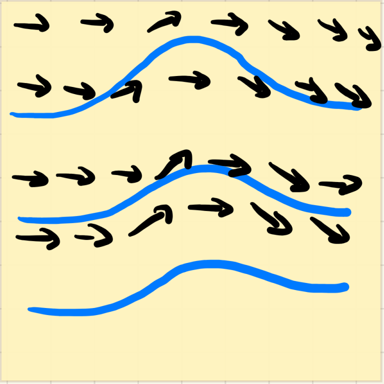
α Vector fields arise in many situations in applied maths. Let us list some:

- The gravitational field $F(x,y,z)$ due to some object is the force of attraction that the object exerts on a unit mass located at position (x,y,z).

- The gradient $\nabla f(x,y,z)$ of any scalar field f gives the direction and magnitude of the greatest rate of increase of f at (x,y,z). In particular, a temperature gradient, $\nabla T(x,y,z)$ is a vector field giving the direction and magnitude of the greatest rate of increase of temperature T at the point (x,y,z) in a heat-conducting medium. Pressure gradients provide similar information about the variation of pressure in a fluid such as an air mass or an ocean.

Field lines (Integral Curves, Trajectories, Streamlines)

A path will be a curve to which the field is tangent at every point. Such curves are called field lines for the given vector field.



The field lines of F do not depend on the magnitude of F at any point but only on the direction of the field. If the field line through some point has parametric equation $r = r(t)$, then its tangent vector dr/dt must be parallel to $F(r(t))$ for all t. Thus,

$$\frac{dr}{dt} = \lambda(t) F(r(t))$$

For some vector fields, this diff. eqn can be integrated to find the field lines.

$$\frac{dx}{dt} = \lambda(t) F_1(x,y,z) \qquad \frac{dy}{dt} = \lambda(t) F_2(x,y,z)$$

$$\frac{dz}{dt} = \lambda(t) F_3(x,y,z)$$

$$\frac{dx}{F_1(x,y,z)} = \frac{dy}{F_2(x,y,z)} = \frac{dz}{F_3(x,y,z)}$$

Example

Find the field lines of $F = xz\,i + 2x^2z\,j + x^2 k$

Soln

The field lines satisfy $\frac{dx}{xz} = \frac{dy}{2x^2z} = \frac{dz}{x^2}$

or equivalently $dy = 2x\,dx$ and $dy = 2z\,dz$

The field lines are the curves of intersection of the two families $y = x^2 + C_1$ and $y = z^2 + C_2$ of parabolic cylinders.

Vector Fields in Polar Coordinates

A vector field in the plane can be expressed in terms of polar coordinates in the form

$$F = F(r, \theta) = F_r(r,\theta)\,\hat{r} + F_\theta(r,\theta)\,\hat{\theta},$$

where $\hat{r}$ and $\hat{\theta}$, defined everywhere except at the origin by

$$\hat{r} = \cos\theta\, i + \sin\theta\, j$$

$$\hat{\theta} = -\sin\theta\, i + \cos\theta\, j,$$

are unit vectors in the direction of increasing r and θ at $[r, \theta]$.

$d\hat{r}/d\theta = \hat{\theta}$ and that $\hat{\theta}$ is just $\hat{r}$ rotated 90° counterclockwise.

A curve with polar equation $r = r(\theta)$ can be expressed in vector parametric form.

$$r = r\hat{r},$$

This curve is a field line of F if its differential tangent vector

$$d\mathbf{r} = dr\,\hat{r} + r\frac{d\hat{r}}{d\theta}d\theta = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$$

is parallel to the field vector $F(r,\theta)$ at any point except the origin, that is, if $r = f(\theta)$ satisfies the differential equation

$$\boxed{\frac{dr}{F_r(r,\theta)} = \frac{r\,d\theta}{F_\theta(r,\theta)}}$$

15.3 Line Integrals

The integrals $\iint_D f(x,y)\,dA$ and $\iiint_R f(x,y,z)\,dV$ represent the total amounts of quantities distributed over regions D in the plane and R in the 3-space in terms of the areal or volume densities of those quantities.

It may happen that a quantity is distributed with specified line density along a curve in the plane or in 3-space, or with specified areal density over a surface in 3-space.

In such cases we require line integrals or surface integrals to add up the contributing elements and calculate the total quantity.

Let $\mathcal{C}$ be a bounded, continuous parametric curve in $\mathbb{R}^3$. That $\mathcal{C}$ is a smooth curve if it has a parametrization of the form

$$r = r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

t in interval I,
with "velocity" vector $v = dr/dt$ continuous and nonzero. We will call C a smooth arc if it is a smooth curve with finite parameter interval $I = [a, b]$.

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

adding up the lengths $|\Delta r_i| = |r_i - r_{i-1}|$ of line segments joining these points, and taking the limit as the maximum distance between adjacent points approached zero. The length was denoted $\int_C ds$ and is a example of a line integral along C having integrand 1.

The line integral of a general function $f(x, y, z)$ can be defined similarly. We choose a point (x_i^*, y_i^*, z_i^*) on the ith subarc and form the Riemann sum

$$\boxed{S_n = \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)|\Delta r_i|}$$

If this sum has a limit as $\max |\Delta r_i| \to 0$, independent of the particular choices of the points (x_i^*, y_i^*, z_i^*), then we call this the line integral of f along C and denote it

$$\boxed{\int_C f(x,y,z)\,ds}$$

Evaluating Line Integrals

The length of C was evaluated by expressing the arc length element $ds = |dr/dt|\,dt$ in terms of a parametrization $r = r(t)$, $(a \le t \le b)$ of the curve, and integrating this from $t=a$ to $t=b$:

$$\text{length of } C = \int_C ds = \int_a^b \left|\frac{dr}{dt}\right| dt$$

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(r(t))\left|\frac{dr}{dt}\right| dt}$$

The value of the line integral of a function f along a curve C depends on f and C but not on the particular way C is parametrized. If $r = r^*(u)$, $\alpha \le u \le \beta$, is another parametrization of the same smooth curve C, then any point $r(t)$ on C can be expressed in terms of the new parametrization as $r^*(u)$, where u depends on t: $u = u(t)$.

If $r^*(u)$ traces C in the same direction as $r(t)$, then $u(a) = \alpha$, $u(b) = \beta$, and $du/dt \ge 0$.

If $r^*(u)$ traces C in the opposite direction then $u(a) = \beta$, $u(b) = \alpha$, and $du/dt \le 0$.

In either event,

$$\int_a^b f(r(t)) \left| \frac{dr}{dt} \right| dt = \int_a^b f(r^*(u(t))) \left| \frac{dr^*}{du} \frac{du}{dt} \right| dt$$

$$= \int_\alpha^\beta f(r^*(u)) \left| \frac{dr^*}{du} \right| du.$$