Improper Integrals and a Mean Value Theorem

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Polar Coordinates and Polar Curves

14.3 – 8.5

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14.3 Improper Integrals and a Mean Value Th.

As in the single-variable case, improper double integrals can arise if either the domain of integration is unbounded or the integrand is unbounded near any point of the domain or its boundary.

Improper Integrals of Positive Functions

An improper integral of a function f satisfying $f(x,y) \geq 0$ on the domain D must either exist or be infinite. Convergence or divergence of improper double integrals of such nonnegative functions can be determined by iterating them and determining the convergence or divergence of any single improper integrals that result

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If $\iint_D |f(x,y)|\,dA$ converges then $\iint_D f(x,y)\,dA$ converges

Such double integrals are called absolutely convergent by analogy with absolutely convergent infinite series.

A mean value theorem for double integrals

Let D be a set in the xy-plane that is closed and bounded has positive are $A = \iint_D dA$. Suppose that $f(x,y)$ is continuous on D. Then there exist points (x_1, y_1) and (x_2, y_2) in D where f assumes minimum and maximum values that is;

$$f(x_1, y_1) \le f(x,y) \le f(x_2, y_2)$$

for all points (x,y) in D. If we integrate this inequality over D, we obtain

$$f(x_1, y_1)\, A = \iint_D f(x_1, y_1)\, dA$$

$$\le \iint_D f(x,y)\, dA \le \iint_D f(x_2, y_2)\, dA = f(x_2, y_2)\, A.$$

Therefore, dividing by A, we find that the number

$$\bar{f} = \frac{1}{A} \iint_D f(x,y)\, dA \rightarrow \left(\begin{array}{c}\text{average value} \\ \text{or} \\ \text{mean value of } f\end{array}\right)$$

lies between the minimum and maximum

values of f on D:

$$f(x_1, y_1) \le \bar{f} \le f(x_2, y_2)$$

A set D in the plane is said to be connected if any two points in it can be joined by a continous parametric curve $x = x(t)$, $y = y(t)$, $(0 \le t \le 1)$, lying in D. Suppose this curve joins (x_1, y_1) (where $t=0$) and (x_2, y_2) (where $t=1$). Let $g(t)$ satisfy

$$g(t) = f(x(t), y(t)), \quad 0 \le t \le 1.$$

Then g is continuous and takes the values $f(x_1, y_1)$ at $t=0$ and $f(x_2, y_2)$ at $t=1$. By the Intermediate-Value Theorem there exists a number t_0 between 0 and 1 such that $\bar{f} = g(t_0) = f(x_0, y_0)$, where $x_0 = x(t_0)$ and $y_0 = y(t_0)$. Thus, we have found a point (x_0, y_0) in D such that

$$\frac{1}{\text{area of } D} \iint_D f(x,y)\, dA = f(x_0, y_0)$$

8.5 Polar Coordinates and Polar Curves

The polar coordinate system is an alternative to the rectangular (cartesian) coordinate system for describing the location of points in a plane. In the polar coordinate system there is an origin (or pole), O, and a polar axis, a ray (i.e. half-line) extending from O horizontally to the right. The position of any point P in the plane is then determined by its polar coordinates $[r, \theta]$, where

(i) r is the distance from O to P, and

(ii) θ is the angle that the ray OP makes with the polar axis (counterclockwise angles being considered positive)

Polar - rectangular conversion

$x = r\cos\theta$ $\qquad x^2+y^2=r^2$

$y = r\sin\theta$ $\qquad \tan\theta = y/x$