Double Integrals

&

Iteration of Double Integrals in Cartesian Coordinates

14.1 - 14.2

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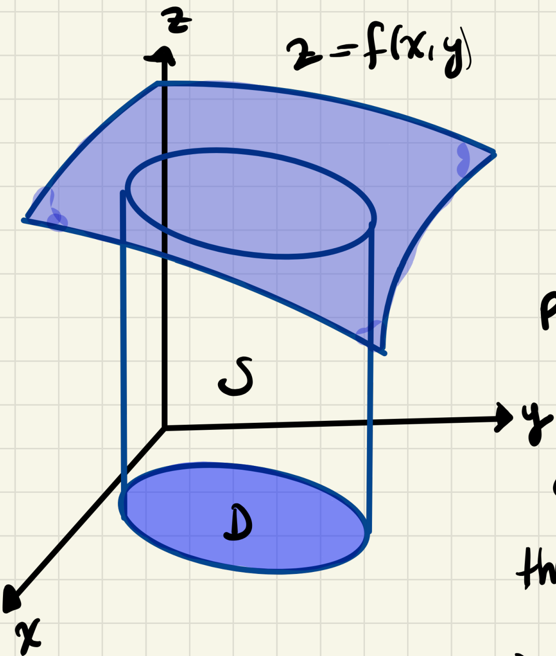
14.1 Double Integrals



A double integral of a function of two variables over a domain D in the plane by means of the standard volume problem of finding the volume of the three-dimensional region S bounded by the surface $z = f(x, y)$ the xy-plane, and the cylinder parallel to the z-axis passing through the boundary of D.

(A solid region S lying above domain D in the xy-plane and below the surface $z = f(x, y)$)

We will define the double integral of $f(x, y)$ over the domain D,

$$\boxed{\iint_D f(x, y)\, dA}$$

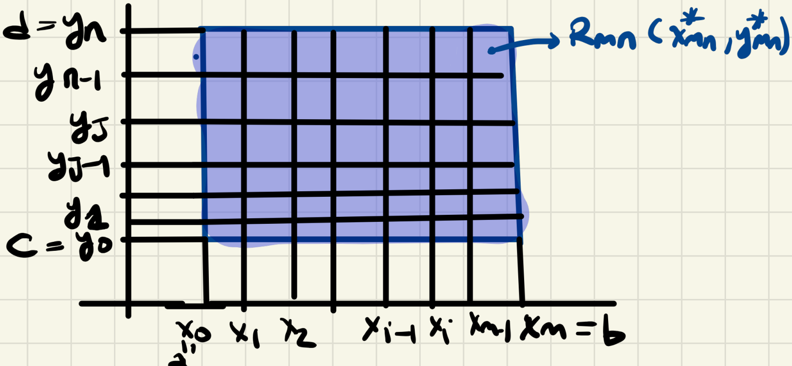
Let D is a closed rectangle with sides parallel to the coordinate axes in the xy-plane, and f is a bounded function on D. If D consists of the points (x, y) st $a \le x \le b$ and $c \le y \le d$, we can form a partition P of D into small rectangles by partitioning each of the intervals $[a, b]$ and $[c, d]$, say by points

$$a = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = b,$$

$$c = y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n = d$$

The partition P of D then consists of the mn rectangles R_{ij} $(1 \le i \le m,\ 1 \le j \le n)$, consisting of points (x, y) for which

$x_{i-1} \le x \le x_i$ and $y_{j-1} \le y \le y_j$



The rectangle R_{ij} has area

$$\Delta A_{ij} = \Delta x_i \, \Delta y_j = (x_i - x_{i-1})(y_j - y_{j-1})$$

and diameter (i.e. diagonal length)

$$\operatorname{diam}(R_{ij}) = \sqrt{(\Delta x_i)^2 + (\Delta y_j)^2}$$

$$= \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}$$

The norm of the partition P is the largest of these subrectangle diameters:

$$\|P\| = \max_{\substack{1 \le i \le m \\ 1 \le j \le n}} \operatorname{diam}(R_{ij}) .$$

Now we pick an arbitrary point (x_{ij}^*, y_{ij}^*) in each of the rectangles R_{ij} and form the Riemann sum

$$R(f, P) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \, \Delta A_{ij} ,$$

which is the sum of mn terms, one for each rectangle in the partition.

Definition 1

The double integral over a rectangle

We say that f is integrable over the rectangle D and has double integral

$$I = \iint_D f(x,y)\,dA,$$

if for every positive number ϵ there exists a number δ depending on ϵ, such that

$$|R(f,P) - I| < \epsilon$$

holds for every partition P of D satisfying $\|P\| < \delta$ and for all choices of the points (x_{ij}^*, y_{ij}^*) in the subrectangles of P.

Definition 2

If $f(x,y)$ is defined and bounded on domain D, let $\hat{f}$ be the extension of f that is zero everywhere outside D:

$$\hat{f}(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \text{ belongs to } D \\ 0, & \text{if } (x,y) \text{ does not belong to } D \end{cases}$$

If D is a bounded domain, then it is contained in some rectangle R with sides parallel to the coordinate axes. If $\hat{f}$ is integrable over R, we say that f is integrable over D and define the double integral of f over D to be

$$\iint_D f(x,y)\,dA = \iint_R \hat{f}(x,y)\,dA$$

Theorem If f is continuous on a closed, bounded domain D whose boundary consists of finitely many curves of finite length, then f is integrable on D.

Properties of the Double Integral

(a) $\iint_D f(x,y)\,dA = 0$ if D has zero area.

(b) Area of a domain: $\iint_D 1\,dA = \text{area of } D$

(c) Integrals representing volumes: If $f(x,y) \geq 0$ on D, then $\iint_D f(x,y)\,dA = V \geq 0$, where V is the volume of the solid lying vertically above D and below the surface $z = f(x,y)$.

(d) If $f(x,y) \leq 0$ on D, then $\iint_D f(x,y)\,dA = -V \leq 0$, where V is the volume of the solid lying vertically below D and above the surface $z = f(x,y)$.

(e) Linear dependence on the integrand:

$$\iint_D (\mathcal{L} f(x,y) + M g(x,y))\,dA = \mathcal{L} \iint_D f(x,y)\,dA + M \iint_D g(x,y)\,dA$$

(f) Inequalities are preserved:

If $f(x,y) \leq g(x,y)$ on D, then $\iint_D f(x,y)\,dA \leq \iint_D g(x,y)\,dA$

(g) The triangle inequality:

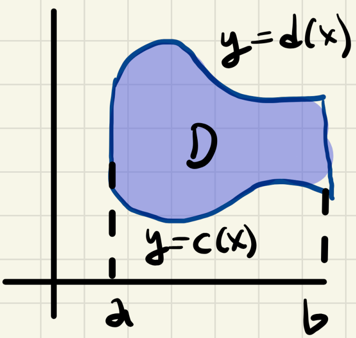
$$\left| \iint_D f(x,y)\,dA \right| \leq \iint_D |f(x,y)|\,dA$$

(h) **Additivity of domains:** If $D_1, D_2, \cdots, D_k$ are nonoverlapping domains on each of which f is integrable, then f is integrable over the union $D = D_1 \cup D_2 \cup \cdots \cup D_k$ and

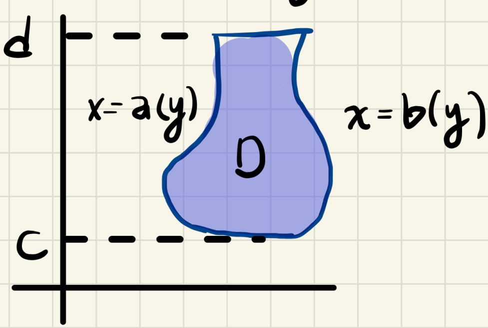
$$\iint_D f(x,y)\,dA = \sum_{j=1}^{k} \iint_{D_j} f(x,y)\,dA$$

14.2 Iteration of Double Integrals in Cartesian Coordinates

The existence of the double integral $\iint_D f(x,y)\,dA$ depends on f and the domain D. Evaluation of double integrals is easiest when the domain of integration is of simple type.

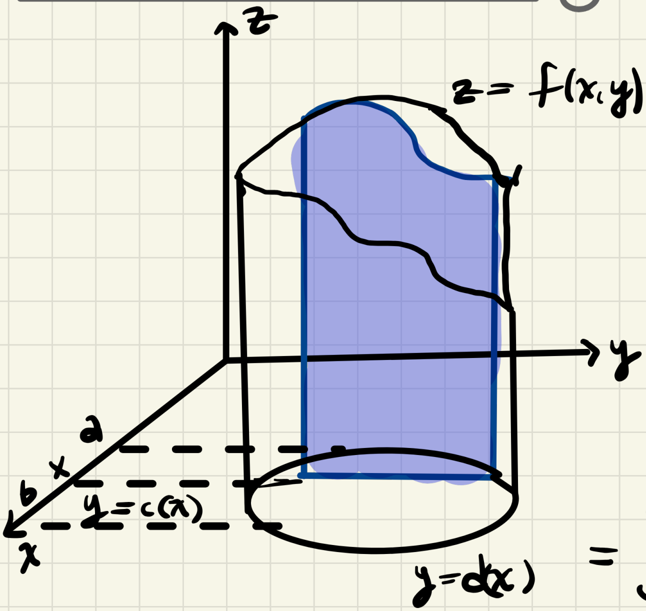


A y-simple domain



An x-simple domain

It can be shown that a bounded, continuous function $f(x,y)$ is integrable over a bounded x-simple or y-simple domain and, therefore, over any regular domain.



$$A(x) = \int_{c(x)}^{d(x)} f(x,y)\,dy$$

$$\iint_D f(x,y)\,dA = \int_a^b A(x)\,dx$$

$$= \int_a^b \left(\int_{c(x)}^{d(x)} f(x,y)\,dy \right) dx$$

$$= \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$$

$$= \int_a^b dx \int_{c(x)}^{d(x)} f(x,y)\,dy$$

Theorem

Iteration of double integrals

If $f(x,y)$ is continuous on the bounded y-simple domain D given by $a \le x \le b$ and $c(x) \le y \le d(x)$, then

$$\boxed{\iint_D f(x,y)\,dA = \int_a^b dx \int_{c(x)}^{d(x)} f(x,y)\,dy}$$

Similarly, if f is continuous on the x-simple domain D given by $c \le y \le d$ and $a(y) \le x \le b(y)$, then

$$\boxed{\iint_D f(x,y)\,dA = \int_c^d dy \int_{a(y)}^{b(y)} f(x,y)\,dx}$$