Cylindrical and Spherical Coordinates

Parametrization and Orientation of Curves

11.1 - 11.3

ORTA DOĞU TEKNİK ÜNİVERSİTESİ

MATEMATİK TOPLULUĞU

11.1 Vector Functions of One Variable

If a particle moves around in 3-space, its motion can be described by giving the three coordinates of its position as functions of time t:

$$x = x(t), \quad y = y(t), \quad \text{and} \quad z = z(t)$$

It is more convenient, however, to replace these three equations by a single vector equation.

$$\mathbf{r} = \mathbf{r}(t)$$

giving the position vector of the moving particle as a function of t. In terms of the standard basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, the position of the particle at time t is

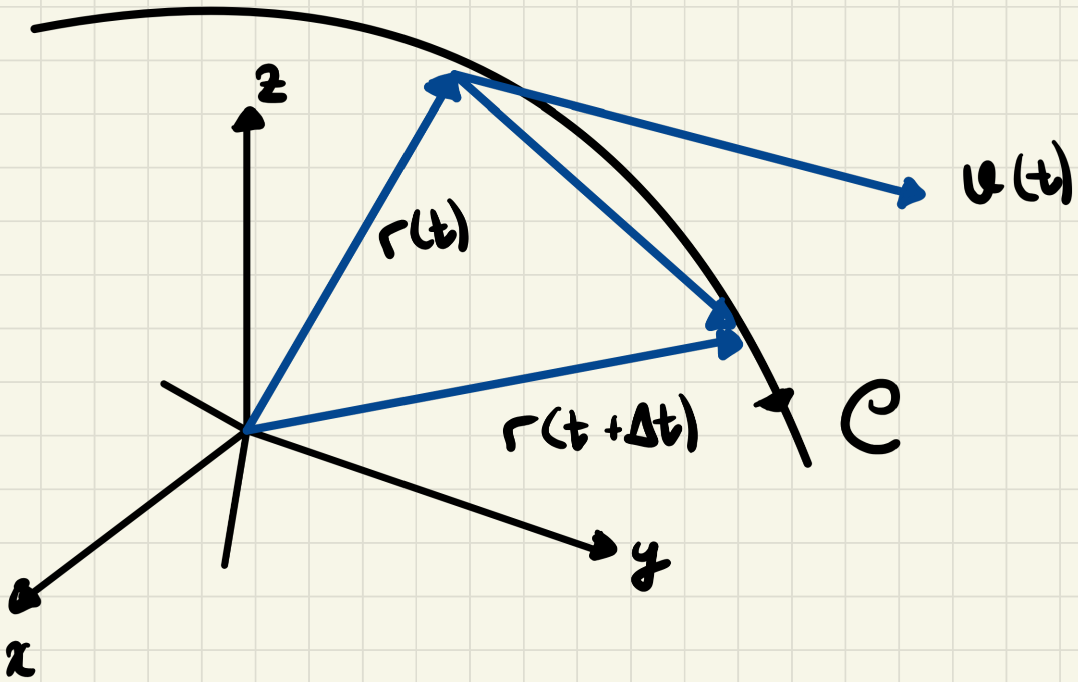
position : $\mathbf{r} = \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$

As t increases, the particle moves along a path, a curve C in 3-space. If $z(t)=0$ then C is a plane curve in the xy-plane. we assume that C is a continuous curve; the particle cannot instantaneously jump from one point to a distant point. This is equivalent to requiring that the component functions $x(t)$, $y(t)$, and $z(t)$ are continuous functions of t, and we therefore say that $r(t)$ is a continuous vector function of t.

In the time interval from t to $t+\Delta t$, the particle moves from position $r(t)$ to position $r(t+\Delta t)$. Therefore, its average velocity is $\boxed{\dfrac{r(t+\Delta t)-r(t)}{\Delta t}}$, which is a vector parallel to the secant vector from $r(t)$ to $r(t+\Delta t)$.

If the average velocity has a limit as $\Delta t \to 0$, then we say that r is differentiable at t, and we call the limit the (instantaneous) velocity of the particle at time t. We denote the velocity vector by $v(t)$:

$$\boxed{\text{velocity}: \quad v(t) = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t} = \frac{d\,r(t)}{dt}}$$



This velocity vector has direction tangent to the path C at the point $r(t)$ and it points in the direction of motion. The length of velocity vector, $\upsilon(t) = |v(t)|$ is called the speed of the particle :

$$\boxed{\text{speed} : \upsilon(t) = |v(t)|.}$$

we define the acceleration of the particle to be the time derivative of the velocity:

$$\boxed{\text{acceleration} : a(t) = \frac{dv}{dt} = \frac{d^2r}{dt^2}}$$

Example

Find the velocity, speed and acceleration, and describe the motion of a particle whose position at time t is

$r = 3\cos\omega t\, i + 4\cos\omega t\, j + 5\sin\omega t\, k$

Solution

$v = dr/dt = -3\omega\sin\omega t\, i - 4\omega\sin\omega t\, j + 5\omega\cos\omega t\, k$

$\upsilon = |v| = 5\omega$

$a = dv/dt = -3\omega^2\cos\omega t\, i - 4\omega^2\cos\omega t\, j - 5\omega^2\sin\omega t\, k$

$= -\omega^2 r$

Theorem 1 (Differentiation rules for vector functions)

Let $u(t)$ and $v(t)$ be differentiable vector-valued functions, and let $\lambda(t)$ be a differentiable scalar-valued function. Then $u(t)+v(t)$, $\lambda(t)u(t)$, $u(t)\cdot v(t)$, $u(t)\times v(t)$, and $u(\lambda(t))$ are differentiable and

(a) $\frac{d}{dt}\left(u(t)+v(t)\right) = u'(t)+v'(t)$

(b) $\frac{d}{dt}\left(\lambda(t)u(t)\right) = \lambda(t)u(t) + \lambda(t)u'(t)$

(c) $\frac{d}{dt}\left(u(t)\cdot v(t)\right) = u'(t)\cdot v(t) + u(t)\cdot v'(t)$

(d) $\frac{d}{dt}\left(u(t)\times v(t)\right) = u'(t)\times v(t) + u(t)\times v'(t)$

(e) $\frac{d}{dt}\left(u(\lambda(t))\right) = \lambda'(t)u'(\lambda(t))$

Also, at any point where $u(t)\neq 0$

(f) $\frac{d}{dt}|u(t)| = \frac{u(t)\cdot u'(t)}{|u(t)|}$

11.3 Curves and Parametrizations

In this section we will consider curves as geometric objects rather than as paths of moving particles. It is difficult to give a formal defn. of a curve as a geometric object without involving the concept of parametric representation.

$$\mathbf{r} = \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \quad a \le t \le b$$

However, the parameter t need no longer represent time or any other specific physical quantity.

Example Use $t = y$ to parametrize the part of the line of intersection of the two planes $y = 2x - 4$ and $z = 3x + 1$ from $(2,0,7)$ to $(3,2,10)$.

Solution Since $y = t$ $\quad x = \frac{1}{2}(y+4) = \frac{1}{2}(t+4)$

$z = 3x + 1 \Rightarrow z = \frac{3}{2}(t+4) + 1 = \frac{3}{2}t + 7$

$\mathbf{r} = \frac{t+4}{2}\,\mathbf{i} + t\,\mathbf{j} + \left(\frac{3}{2}t + 7\right)\mathbf{k}, \quad 0 \le t \le 2.$

The curve $r = r(t)$, $(a \leq t \leq b)$, is called a closed curve if $r(a) = r(b)$.

The curve C is non-self-intersecting if there exists some parametrization $r = r(t)$, $(a \leq t \leq b)$, of C that is one-to-one except that the endpoints could be the same:

$$r(t_1) = r(t_2) \quad a \leq t_1 < t_2 \leq b \Rightarrow t_1 = a \text{ and } t_2 = b$$

Such a curve can be closed, but otherwise does not intersect itself; it is then called a simple closed curve.

Parametrizing the Curve of Intersection of Two Surfaces

There is no unique way to do this, but if one of the given surfaces is a cylinder parallel to a coordinate axis, we can begin by parametrizing that surface.

Example Parametrise the curve of intersection of the plane $x+2y+4z=4$ and the elliptic cylinder $x^2+4y^2=4$.

Solution

We begin with the eqn. $x^2+4y^2=4$, which is independent of z. It can be parametrized in many ways; one convenient way is

$$x = 2\cos t, \quad y = \sin t, \quad (0 \le t \le 2\pi)$$

The eqn of the plane can then be solved for z, so that z can be expressed in terms of t:

$$z = \frac{1}{4}(4-x-2y) = 1 - \frac{1}{2}(\cos t + \sin t)$$

Thus, the given surfaces intersect in the curve

$$\mathbf{r} = 2\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \left(1 - \frac{\cos t + \sin t}{2}\right)\mathbf{k}$$

$$(0 \le t \le 2\pi)$$