SOLUTION KEY		
FULL NAME	STUDENT ID	DURATION: 100 MINUTES 5 QUESTIONS ON 4 PAGES TOTAL: 60 POINTS

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Q1) (12 *pts*)Let u, v, w be vectors in a vector space V and let W be the subspace spanned by $\{u, v, w\}$ (i.e., $W = \langle u, v, w\rangle$). If $\dim(W) = 3$. Show that

$$\{u + v, v + w, w + u\}$$

is a basis of W.

Let $B = \{u + v, v + w, w + u\}$.
Since $\dim(W) = 3$ and B contains 3 vectors, if we can show show that B is linearly independent then B will also span W.

Simlarly, since $dim(W) = 3$ the set $\{u, v, w\}$ is a basis of W.

So, Suppose $c_1(u + v) + c_2(v + w) + c_3(w + u) = 0$ for some scalars c_1, c_2, c_3.

This implies, $(c_1 + c_3)u + (c_1 + c_2)v + (c_2 + c_3)w = 0$.

Since u, v, w are linearly independent we have $c_1 + c_3 = c_1 + c_2 = c_2 + c_3 = 0$.

Hence c_1, c_2, c_3 are a solution to the system

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing, we have :

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

which show that the system has only the trivial solution. Thus $c_1 = c_2 = c_3 = 0$ and hence B is a basis of W.

Q2) $(1=+5=15\ pts)$ Let A be a 4×5 matrix, which is row equivalent to the matrix

$$B=\begin{bmatrix}1&1&1&1&1\\0&0&1&1&1\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}$$

by the row operations:

$\mathcal{E}_1=-2R_1+R_2\quad \mathcal{E}_2=-3R_1+R_2\quad \mathcal{E}_3=-3R_1+R_3\quad \mathcal{E}_4=-3R_1+R_4\quad \mathcal{E}_5=-R_3+R_4\quad \mathcal{E}_6=R_2\leftrightarrow R_3\quad \mathcal{E}_7=R_3\leftrightarrow R_4$

That is $B=\mathcal{E}_7\mathcal{E}_6\mathcal{E}_5\mathcal{E}_4\mathcal{E}_3\mathcal{E}_2\mathcal{E}_1(A)$

(a) Find the fundamental solutions of the homogeneous system $AX=0$ and express the general solution as a linear combination of the fundamental solutions.

Since A and B are row equivalent, $AX=0$ and $BX=0$ have the same solutions.
Since B is in echelon form, letting x_1,x_3,x_5 be basic variables and x_2,x_4 free variables, we have $x_1+x_2+x_3+x_4+x_5=0\ ,\quad x_3+x_4+x_5=0\quad ,\quad x_5=0$ and hence

$$x_5=0,x_3=-x_4,x_1=-x_2$$

Thus, the general solution is of the form
$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}-x_2\\x_2\\-x_4\\x_4\\0\end{bmatrix}=x_2\begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\-1\\1\\0\end{bmatrix}$, $x_2,x_4\in\mathbb{R}$ with fundamental solutions $\begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\-1\\1\\0\end{bmatrix}$

(b) Determine $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ so that $AX=\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ has a solution.

Applying the row operations to the matrix $K=\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ we get $K'=\begin{bmatrix}a\\-3a+c\\d-c\\-5a+b\end{bmatrix}$. Thus, $AX=K$ is equivalent to $BX=K'$ which has a solution if and only if $b=5a$.

Q3) $(9+3=12\ pts)$ Let

$$A=\begin{bmatrix}1 & 0 & 2b\\ 1 & a+2 & b\\ -2 & a+2 & -4b\end{bmatrix}$$

(a) Find the values of a and b such that A is an invertible matrix.

$$\begin{bmatrix}1 & 0 & 2b\\ 1 & a+2 & b\\ -2 & a+2 & -4b\end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix}1 & 0 & 2b\\ 0 & a+2 & -b\\ -2 & a+2 & -4b\end{bmatrix} \xrightarrow{2R_1+R_3} \begin{bmatrix}1 & 0 & 2b\\ 0 & a+2 & -b\\ 0 & a+2 & 0\end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix}1 & 0 & 2b\\ 0 & a+2 & -b\\ 0 & 0 & b\end{bmatrix}$$

$$\xrightarrow{R_3+R_2} \begin{bmatrix}1 & 0 & 2b\\ 0 & a+2 & 0\\ 0 & 0 & b\end{bmatrix} \xrightarrow{-2R_3+R_1} \begin{bmatrix}1 & 0 & 0\\ 0 & a+2 & 0\\ 0 & 0 & b\end{bmatrix}$$

So, A is r.e. to the identity matrix if and only if $a \neq -2$ and $b \neq 0$

(b) Find the values of a and b such that $AX=0$ has infinitely many solutions.

$AX=0$ has infinitely many solutions if and only if A is not invertible. By part a), A is not invertible if and only if $a=-2$ or $b=0$

Q4) $(3+3=6\ pts)$ For the following statements, either show that they are true or give a counter example.

(a) Let A, B and C be $n \times n$ matrices. If A is row-equivalent to B, then AC is row-equivalent BC.

This statement is TRUE:

A is r.e. to B if and only if there is an invertible matrix P such that $B=PA$. Hence, $BC=P(AC)$, which implies that AC is r.e. to BC

(b) The set $W=\left\{(x,y,z)\in\mathbb{R}^3 \,|\, x^2+y^2=z\right\}$ is a subspace of $\mathbb{R}^3$.

This statement is FALSE:

The element $(1,1,2)$ is in W, but $2(1,1,2)$ is not in W. Hence W is not closed under scalar multiplication and thus is not a subspace.

Q5) $(5+10=15\ pts)$ Let V be the vector space of $n\times n$ matrices. Let $W\subseteq V$ be the subset consisting of skew-symmetric matrices, that is, $W=\{A\in V\mid A^T=-A\}$.

(a) Show that W is a subspace of V.

Clearly, the zero matrix is in W, and hence $W\neq\emptyset$.

Let $A,B\in W$. Then $A=-A^T$ and $B=-B^T$. So $(A+B)^T=A^T+B^T=-A-B=-(A+B)$. Hence $A+B\in W$.

Let $c\in\mathbb{R}$ and $A\in W$. Then $(cA)^T=c(A^T)=c(-A)=-(cA)$. Hence $cA\in W$.

Therefore, W is a subspace of V.

(b) For $n=3$, find a basis for W and determine $\dim(W)$.

Let

$$B=\left\{M_1=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix},M_2=\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix},M_3=\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}\right\}\subseteq W$$

We claim B is a basis for W.

Let $A=\begin{bmatrix}0&a&b\\-a&0&c\\-b&-c&0\end{bmatrix}\in W$ where $a,b,c\in\mathbb{R}$. Then, we have

$$A=aM_1+bM_2+cM_3$$

Hence, B spans W.

Suppose $c_1M_1+c_2M_2+c_3M_3=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ for some $c_1,c_2,c_2\in\mathbb{R}$.

Then, $\begin{bmatrix}0&c_1&c_2\\-c_1&0&c_3\\-c_2&-c_3&0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ which implies $c_1=c_2=c_3=0$.

This shows B is linearly independent. Hence $dim(W)=3$.