

SOLUTION KEY		
F U L L N A M E	S T U D E N T I D	DURATION: 100 MINUTES 5 QUESTIONS ON 4 PAGES TOTAL: 60 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature

Q1) (12 pts) Let u, v, w be vectors in a vector space V and let W be the subspace spanned by $\{u, v, w\}$ (i.e., $W = \langle u, v, w \rangle$). If $\dim(W) = 3$. Show that

$$\{u + v, v + w, w + u\}$$

is a basis of W .

Since $\dim(W) = 3$ and B contains 3 vectors, it is enough to show that B is linearly independent.

So, Suppose $c_1(u + v) + c_2(v + w) + c_3(w + u) = 0$ for some scalars c_1, c_2, c_3 .

This implies, $(c_1 + c_3)u + (c_1 + c_2)v + (c_2 + c_3)w = 0$.

Since u, v, w are linearly independent we have $c_1 + c_3 = c_1 + c_2 = c_2 + c_3 = 0$.

Hence c_1, c_2, c_3 are a solution to the system

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing, we have :

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

which show that the system has only the trivial solution. Thus $c_1 = c_2 = c_3 = 0$.

Q2) ($1 = +5 = 15$ pts) Let A be a 4×5 matrix, which is row equivalent to the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

by the row operations:

$$\mathcal{E}_1 = -2R_1 + R_2 \quad \mathcal{E}_2 = -3R_1 + R_2 \quad \mathcal{E}_3 = -3R_1 + R_3 \quad \mathcal{E}_4 = -3R_1 + R_4 \quad \mathcal{E}_5 = -R_3 + R_4 \quad \mathcal{E}_6 = R_2 \leftrightarrow R_3 \quad \mathcal{E}_7 = R_3 \leftrightarrow R_4$$

That is $B = \mathcal{E}_7 \mathcal{E}_6 \mathcal{E}_5 \mathcal{E}_4 \mathcal{E}_3 \mathcal{E}_2 \mathcal{E}_1(A)$

(a) Find the fundamental solutions of the homogeneous system $AX = 0$ and express the general solution as a linear combination of the fundamental solutions.

Since A and B are row equivalent, $AX = 0$ and $BX = 0$ have the same solutions.

Since B is in echelon form, letting x_1, x_3, x_5 be basic variables and x_2, x_4 free variables, we have $x_1 + x_2 + x_3 + x_4 + x_5 = 0$, $x_3 + x_4 + x_5 = 0$, $x_5 = 0$ and hence

$$x_5 = 0, x_3 = -x_4, x_1 = -x_2$$

Thus, the general solution is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ -x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, x_2, x_4 \in \mathbb{R} \text{ with fundamental solutions } \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

(b) Determine $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ so that $AX = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ has a solution.

Applying the row operations to the matrix $K = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ we get $K' = \begin{bmatrix} a \\ -3a + c \\ d - c \\ -5a + b \end{bmatrix}$. Thus, $AX = K$ is equivalent to $BX = K'$ which has a solution if and only if $b = 5a$.

Q3) (9 + 3 = 12 pts) Let

$$A = \begin{bmatrix} 1 & 0 & 2b \\ 1 & a+2 & b \\ -2 & a+2 & -4b \end{bmatrix}$$

(a) Find the values of a and b such that A is an invertible matrix.

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 2b \\ 1 & a+2 & b \\ -2 & a+2 & -4b \end{bmatrix} &\xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 0 & 2b \\ 0 & a+2 & -b \\ -2 & a+2 & -4b \end{bmatrix} \xrightarrow{2R_1+R_3} \begin{bmatrix} 1 & 0 & 2b \\ 0 & a+2 & -b \\ 0 & a+2 & 0 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & 0 & 2b \\ 0 & a+2 & -b \\ 0 & 0 & b \end{bmatrix} \\ \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 0 & 2b \\ 0 & a+2 & 0 \\ 0 & 0 & b \end{bmatrix} \xrightarrow{-2R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a+2 & 0 \\ 0 & 0 & b \end{bmatrix} \end{aligned}$$

So, A is r.e. to the identity matrix if and only if $a \neq -2$ and $b \neq 0$

(b) Find the values of a and b such that $AX = 0$ has infinitely many solutions.

$AX = 0$ has infinitely many solutions if and only if A is not invertible. By part a), A is not invertible if and only if $a = -2$ or $b = 0$

Q4) (3 + 3 = 6 pts) For the following statements, either show that they are true or give a counter example.

(a) Let A, B and C be $n \times n$ matrices. If A is row-equivalent to B , then AC is row-equivalent to BC .

This statement is TRUE:

A is r.e. to B if and only if there is an invertible matrix P such that $B = PA$. Hence, $BC = P(AC)$, which implies that AC is r.e. to BC

(b) The set $W = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z\}$ is a subspace of \mathbb{R}^3 .

This statement is FALSE:

The element $(1, 1, 2)$ is in W , but $2(1, 1, 2)$ is not in W . Hence W is not closed under scalar multiplication and thus is not a subspace.

Q5) (5 + 10 = 15 pts) Let V be the vector space of $n \times n$ matrices. Let $W \subseteq V$ be the subset consisting of skew-symmetric matrices, that is, $W = \{A \in V \mid A^T = -A\}$.

(a) Show that W is a subspace of V .

Clearly, the zero matrix is in W , and hence $W \neq \emptyset$.

Let $A, B \in W$. Then $A = -A^T$ and $B = -B^T$. So $(A+B)^T = A^T + B^T = -A - B = -(A+B)$. Hence $A+B \in W$.

Let $c \in \mathbb{R}$ and $A \in W$. Then $(cA)^T = c(A^T) = c(-A) = -(cA)$. Hence $cA \in W$.

Therefore, W is a subspace of V .

(b) For $n = 3$, find a basis for W and determine $\dim(W)$.

Let

$$B = \left\{ M_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\} \subseteq W$$

We claim B is a basis for W .

Let $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \in W$ where $a, b, c \in \mathbb{R}$. Then, we have

$$A = aM_1 + bM_2 + cM_3$$

Hence, B spans W .

Suppose $c_1M_1 + c_2M_2 + c_3M_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for some $c_1, c_2, c_3 \in \mathbb{R}$.

Then, $\begin{bmatrix} 0 & c_1 & c_2 \\ -c_1 & 0 & c_3 \\ -c_2 & -c_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which implies $c_1 = c_2 = c_3 = 0$.

This shows B is linearly independent.