M E T U Department of Mathematics

Math 219 Introduction to Differential Equations Fall 2024-25 Midterm 2 21 Dec. 2024 13:30		
FULL NAME	STUDENT ID	DURATION 120 MINUTES
6 QUESTIONS ON 4 PAGES	SHOW ALL YOUR WORK	TOTAL 100 POINTS

***(15 pts)* 1.** Consider the non-homogeneous system of first order ODE's

$$\begin{aligned} x' &= y \\ y' &= -x + \frac{1}{\sin(t)}, \qquad 0 < t < \pi. \end{aligned}$$

Given that the solutions of the associated homogeneous system are $\begin{bmatrix} x \\ y \end{bmatrix}_h = c_1 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$ for $c_1, c_2 \in \mathbb{R}$, find the general solution of the non-homogeneous system.

$\Psi(t) = \begin{bmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{bmatrix}$ is a fundamental matrix.

$\det(\Psi(t)) = -\sin^2(t) - \cos^2(t) = -1$

$$\Psi^{-1} = \begin{bmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{bmatrix}$$

$\begin{bmatrix} x \\ y \end{bmatrix} = \Psi \int \Psi^{-1} \vec{b}\, dt$ where $\vec{b} = \begin{bmatrix} 0 \\ \frac{1}{\sin(t)} \end{bmatrix}$

$$\Psi^{-1}\vec{b} = \begin{bmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sin(t)} \end{bmatrix} = \begin{bmatrix} \frac{\cos(t)}{\sin(t)} \\ -1 \end{bmatrix}$$

$$\int \Psi^{-1}\vec{b}\, dt = \begin{bmatrix} \int \frac{\cos(t)}{\sin(t)} dt \\ \int -1\, dt \end{bmatrix} = \begin{bmatrix} \ln|\sin(t)| + c_1 \\ -t + c_2 \end{bmatrix}$$

$|\sin(t)| = \sin(t)$ for $0 < t < \pi$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \Psi \int \Psi^{-1}\vec{b}\, dt = \begin{bmatrix} \sin(t) \cdot \ln(\sin(t)) - t\cos(t) \\ \cos(t) \cdot \ln(\sin(t)) + t\sin(t) \end{bmatrix} + c_1 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

$c_1, c_2 \in \mathbb{R}$.

(20 pts) **2.** Solve the initial value problem $y'' + y' - 20y = 2e^{4t}$, $y(0) = 0$, $y'(0) = 1$ by using the method of undetermined coefficients.

First solve $y'' + y' - 20y = 0 \rightsquigarrow$ char. eqn. $r^2 + r - 20 = 0$

$(r+5)(r-4) = 0 \quad r_1 = -5, r_2 = 4$

So, $y_h = c_1 e^{-5t} + c_2 e^{4t}$.

For y_p, try $y_p(t) = Ate^{4t}$ (repetition for $r=4$)

$y_p' = Ae^{4t} + 4Ate^{4t}$

$y_p'' = 8Ae^{4t} + 16Ate^{4t}$

$y_p'' + y_p' - 20y_p = 8Ae^{4t} + \cancel{16Ate^{4t}} + Ae^{4t} + \cancel{4Ate^{4t}} - \cancel{20Ate^{4t}}$

$= 9Ae^{4t} = 2e^{4t} \Rightarrow A = 2/9$

$y = y_h + y_p = c_1 e^{-5t} + c_2 e^{4t} + \frac{2}{9} te^{4t}$.

$y(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$y' = -5c_1 e^{-5t} + 4c_2 e^{4t} + \frac{2}{9} e^{4t} + \frac{8}{9} te^{4t}$

$y'(0) = -5c_1 + 4c_2 + \frac{2}{9} = 1$

$9c_2 = 7/9 \Rightarrow c_2 = 7/81 \quad , c_1 = -7/81$

$$\boxed{y(t) = -\frac{7}{81} e^{-5t} + \frac{7}{81} e^{4t} + \frac{2}{9} te^{4t}}$$

(10 pts) **3.** Find a linear, constant coefficient, homogeneous ODE of the lowest possible order such that both of $y_1(t) = e^{3t} + \cos(2t)$ and $y_2(t) = e^{3t} - \cos(2t)$ are its solutions.

By superposition, $\underline{\frac{y_1 + y_2}{2}} = e^{3t}$ and $\underline{\frac{y_1 - y_2}{2}} = \cos(2t)$ are also solutions of this ODE.

Hence, its characteistic equation must contain the factors $(r-3)$ and (r^2+4).

The minimal order of such an ODE is 3, with char. eqn. $(r-3)(r^2+4) = 0$

$(r-3)(r^2+4) = r^3 - 3r^2 + 4r - 12$

$$\Rightarrow \boxed{y''' - 3y'' + 4y' - 12y = 0}$$

(5+10+5=20 pts) **4.** Suppose that an object of mass $m = 2kg$ is attached to a linear spring with spring constant $k = 20kg/s^2$ in a medium with damping coefficient $\gamma = 4kg/s$. An external force of $F(t) = 10\cos(\omega t)\ kg \cdot m/s^2$ is applied on the object where ω is constant.

(a) Write a differential equation for the position $x(t)$ of the object at time t. Is this system overdamped, critically damped or underdamped?

$$2x'' + 4x' + 20x = 10\cos(\omega t) \Rightarrow x'' + 2x' + 10x = 5\cos(\omega t)$$

$\Delta = 4^2 - 4 \cdot 2 \cdot 20 = -144 < 0$. So, the system is underdamped.

(b) Find the general solution of this differential equation and write it in the form $x(t) = x_p(t) + x_h(t)$ with $x_p(t) = A\cos(\omega t) + B\sin(\omega t)$ and $\lim_{t\to+\infty} x_h(t) = 0$. (In your answer, A and B should be expressed in terms of ω).

$$x'' + 2x' + 10x = 0 \rightarrow r^2 + 2r + 10 = 0$$

$$(r+1)^2 + 3^2 = 0 \Rightarrow r_{1,2} = -1 \mp 3i$$

$$x_h(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)$$

$$x_p(t) = A\cos(\omega t) + B\sin(\omega t) \Rightarrow x_p'(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$$

$$x_p''(t) = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$$

$$x_p'' + 2x_p' + 10x_p = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t) + 2(-\omega A\sin(\omega t) + \omega B\cos(\omega t)) + 10(A\cos(\omega t) + B\sin(\omega t))$$

$$= (2\omega B + (10-\omega^2)A)\cos(\omega t)) + (-2\omega A + (10-\omega^2)B)\sin(\omega t))$$

So, $2\omega B + (10-\omega^2)A = 5$

$-2\omega A + (10-\omega^2)B = 0 \rightarrow A = \frac{10-\omega^2}{2\omega}B$

$$\Rightarrow \left(2\omega + \frac{(10-\omega^2)^2}{2\omega}\right)B = 5 \Rightarrow B = 5 \cdot \frac{2\omega}{(10-\omega^2)^2 + (2\omega)^2},\quad A = 5\frac{10-\omega^2}{(10-\omega^2)^2 + (2\omega)^2}$$

(c) Find the value of ω which maximizes the amplitude of $x_p(t)$.

$$R^2 = A^2 + B^2 = \frac{5^2\left((2\omega)^2 + (10-\omega^2)^2\right)}{\left((2\omega)^2 + (10-\omega^2)^2\right)^2} = \frac{5^2}{(2\omega)^2 + (10-\omega^2)^2}.$$

We should maximize R^2, hence minimize its denominator $(10-\omega^2)^2 + (2\omega)^2 = \omega^4 - 16\omega^2 + 100$.

This happens when $\omega^2 = 8$, namely, $\boxed{\omega = 2\sqrt{2}}$

(15 pts) **5.** Find the general solution of the differential equation $8(x-1)^2y'' + 2(x-1)y' + y = 0, \quad x > 1$.

Try a solution of the form $y = (x-1)^r$.

$y' = r(x-1)^{r-1}, \quad y'' = r(r-1)(x-1)^{r-2}$.

$$8(x-1)^2 r(r-1)(x-1)^{r-2} + 2(x-1)r(x-1)^{r-1} + (x-1)^r = 0$$

$$(8r(r-1) + 2r + 1)(x-1)^r = 0$$

We need $8r(r-1) + 2r + 1 = 0$, hence

$8r^2 - 6r + 1 = 0$. $\quad r_{1,2} = \frac{6 \mp \sqrt{36-32}}{16} = \frac{6 \mp 2}{16}$

$r_1 = \frac{1}{2}, \quad r_2 = \frac{1}{4}$.

$$\boxed{y = c_1(x-1)^{1/2} + c_2(x-1)^{1/4}}$$

(4+10+6=20 pts) **6.** Consider the initial value problem

$$(1+x^2)y'' + y = 0, \qquad y(0) = 1, y'(0) = -1.$$

(a) Find all ordinary points and all singular points of the differential equation in the complex plane.

$y'' + \frac{1}{1+x^2} y = 0$. 0 and $\frac{1}{1+x^2}$ are rational functions. So they are analytic wherever they are continuous. Hence the only singularities are $\mp i$, all other points are ordinary.

(b) Find the recursion relation for a power series solution of the ODE centered at $x_0 = 0$ and calculate the first 5 non-zero terms of the solution of the IVP.

$y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$(1+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$n=0$, x^0: $2a_2 + a_0 = 0 \Rightarrow a_2 = -a_0/2$

$n=1$, x^1: $3\cdot 2\cdot a_3 + a_1 = 0 \Rightarrow a_3 = -a_1/6$

$n \geq 2$, x^n: $(n+2)(n+1)a_{n+2} + (n(n-1)+1)a_n = 0$

$$\Rightarrow \boxed{a_{n+2} = -\frac{n^2-n+1}{n^2+3n+2} a_n}$$

$y(0) = 1 \Rightarrow a_0 = 1$, $y'(0) = -1 \Rightarrow a_1 = -1$.

$a_2 = -\frac{1}{2}$, $a_3 = \frac{1}{6}$, $a_4 = -\frac{3}{12}\cdot a_2 = \frac{1}{8}$.

$y = 1 - x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \dots$

(c) Determine a lower bound for the radius of convergence of the power series solutions of this ODE centered at $x_0 = 0$.

The distance from $x_0 = 0$ to the nearest singularity ($\mp i$) is 1.

Hence, the power series solution above has a radius of convergence $\rho \geq 1$.