METU
Department of Mathematics

	Introduction to Differential Equations Final Exam	

Code: *Math 219* Semester: *Fall 2018* Date: *10 January 2019* Time: *17:00* Duration: *140 minutes*	Last Name: Name: Department: Student No.: Signature: 7 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	SHOW YOUR WORK	

Question 1 (10 pts) Find all solutions of the differential equation

$$y' = \frac{y + xe^{-y/x}}{x}.$$

$y' = \frac{y}{x} + e^{-y/x}$. The equation is homogeneous.

Let $v = y/x$. Then $y' = v + x \cdot v'$

$$\cancel{v} + x \cdot v' = \cancel{v} + e^{-v}$$

$$\int e^v dv = \int \frac{dx}{x}$$

$$e^v = \ln|x| + c$$

$$v = \ln(\ln|x| + c)$$

$$\boxed{y = x \cdot \ln(\ln|x| + c)}$$

Question 2 (10 pts) Write a suitable form for a particular solution of the differential equation

$$y^{(4)} + 2y'' + y = t\cos(t) + te^{2t}$$

with at most 6 coefficients, if the method of undetermined coefficients is to be used. Do not evaluate the coefficients.

$$\underbrace{(D^4 + 2D^2 + 1)}_{(D^2+1)^2} y = t\cos(t) + te^{2t}$$

Annihilator: $(D^2+1)^2$ Annihilator: $(D-2)^2$

Therefore, $(D^2+1)^4(D-2)^2 y = 0$ for any solution. Excluding the terms appearing already in y_h, we get

$$\boxed{y_p = c_1 t^2 \cos t + c_2 t^2 \sin t + c_3 t^3 \cos t + c_4 t^3 \sin t + c_5 e^{2t} + c_6 te^{2t}}$$

Question 3 (20 pts) Find all solutions of the following 2×2 non-homogeneous system by using variation of parameters:

$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix}, \qquad 0 < t < \pi.$$

$$\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda)+5 = \lambda^2+1$$

$\Rightarrow$ eigenvalues are $\lambda_{1,2} = \mp i$

Eigenvectors for $\lambda_1 = +i$:

$$\left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2-i & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{r} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} k$$

$$\vec{z} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} e^{it} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)(\cos t + i \sin t)$$

$$= \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$$

$$\vec{x}^{(1)} = \mathrm{Re}(\vec{z}) = \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix}, \quad \vec{x}^{(2)} = \mathrm{Im}(\vec{z}) = \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$$

$$\Psi(t) = \left[\vec{x}^{(1)} \,|\, \vec{x}^{(2)} \right] = \begin{bmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{bmatrix}. \quad \det \Psi = -1$$

$$\Psi^{-1}(t) = \begin{bmatrix} -\sin t & \cos t + 2\sin t \\ \cos t & -2\cos t + \sin t \end{bmatrix}, \quad \Psi^{-1}\vec{b} = \begin{bmatrix} \cos^2 t + 2\sin t\cos t \\ -2\cos^2 t + \sin t \cos t \end{bmatrix}$$

$$\vec{x} = \Psi \int \Psi^{-1} \vec{b}\, dt = \Psi \int \begin{bmatrix} \frac{1+\cos 2t}{2} + \sin 2t \\ -1 - \cos 2t + \frac{\sin 2t}{2} \end{bmatrix} dt$$

$$= \Psi \left(\begin{bmatrix} \frac{t}{2} + \frac{\sin 2t}{4} - \frac{\cos 2t}{2} + c_1 \\ -t - \frac{\sin 2t}{2} - \frac{\cos 2t}{4} + c_2 \end{bmatrix} \right)$$

Question 4 (10 pts) Suppose that $\mathcal{L}\{g(t)\} = G(s)$. Express $\mathcal{L}^{-1}\left\{ \frac{G(s)}{s^2+4} \right\}$ as a convolution integral in terms of $g(t)$.

$\mathcal{L}\{g(t)\} = G(s)$ and $\mathcal{L}\left\{ \frac{\sin(2t)}{2} \right\} = \frac{1}{s^2+4}$

$\Rightarrow$ By convolution theorem,

$$\mathcal{L}^{-1}\left\{ \frac{G(s)}{s^2+4} \right\} = \frac{1}{2} g(t) * \sin(2t)$$

$$\boxed{= \frac{1}{2} \int_0^t g(\tau) \sin(2(t-\tau))\, d\tau}$$

Question 5 (20 pts) By using the Laplace transform, solve the initial value problem

$$y'' + y = f(t) + \delta(t-3), \quad y(0) = y'(0) = 0$$

for $t \geq 0$ where $\delta(t)$ denotes the impulse function and $f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -1, & 1 \leq t < 2, \\ 0, & 2 \leq t. \end{cases}$

$$f(t) = u_0(t) - 2u_1(t) + u_2(t)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{\delta(t-3)\}$$

$$(s^2+1)Y(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} + e^{-3s}$$

$$Y(s) = \frac{1 - 2e^{-s} + e^{-2s}}{s(s^2+1)} + \frac{e^{-3s}}{s^2+1}$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \Rightarrow 1 = (A+B)s^2 + C \cdot s + A$$

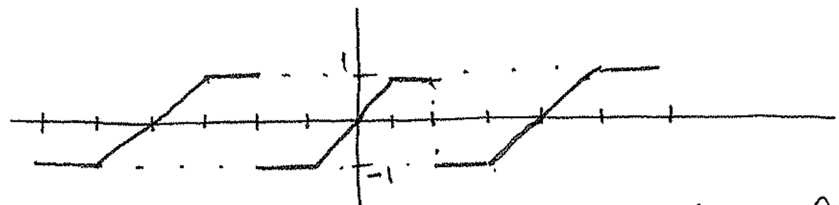
$$\Rightarrow A = 1,\ B = -1,\ C = 0.$$

$$= \frac{1}{s} - \frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t$$

$$\boxed{y(t) = (1-\cos t) - 2u_1(t) \cdot (1 - \cos(t-1)) + u_2(t) \cdot (1 - \cos(t-2)) + u_3(t) \cdot \sin(t-3)}$$

Question 6 (10 pts) Suppose that $f(x) = \begin{cases} x, & 0 \leq x < 1, \\ 1, & 1 \leq x < 2. \end{cases}$ Sketch the graph of the odd extension of $f(x)$ as a periodic function with period 4, for $-6 \leq x \leq 6$. Find the Fourier series expansion of this odd extension.



Period $= 2L = 4$
$\Rightarrow L = 2$

Since the extension is odd, only sin terms will appear.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x) \cdot \sin\left(\frac{n\pi x}{2}\right) dx = \frac{2}{2}\int_0^2 f(x) \cdot \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_0^1 \underbrace{x}_{u} \cdot \underbrace{\sin\left(\frac{n\pi x}{2}\right) dx}_{dv} + \int_1^2 \sin\left(\frac{n\pi x}{2}\right) dx$$

$$\left(\begin{array}{l} du = dx \\ v = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) \end{array}\right)$$

$$= x \cdot \left(\frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right)\Big|_0^1 - \int_0^1 \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) dx - \frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_1^2$$

$$= \frac{-2}{n\pi}\cancel{\cos\left(\frac{n\pi}{2}\right)} + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\cos(n\pi) + \cancel{\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)} = \begin{cases} \frac{4}{n^2\pi^2} + \frac{2}{n\pi}, & n \equiv 1 \pmod 4 \\ \frac{-4}{n^2\pi^2} + \frac{2}{n\pi}, & n \equiv 3 \pmod 4 \\ -2/n\pi, & n \equiv 0, 2 \pmod 4 \end{cases}$$

Question 7 (20 pts) Find the solution of the heat equation below subject to the conditions:

$$\begin{aligned} u_{xx} &= u_t, \quad 0 < x < \pi, \ t > 0 \\ u_x(0,t) &= u_x(\pi,t) = 0, \quad t > 0 \\ u(x,0) &= 1 - \cos(4x), \quad 0 < x < \pi \end{aligned}$$

Show all steps: Apply separation of variables, solve the resulting two point boundary value problem, and find the solution $u(x,t)$.

$u(x,t) = X(x) \cdot T(t)$. $\quad u_{xx} = u_t \Rightarrow X'' \cdot T = X \cdot T'$.

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

$X'' + \lambda X = 0$, $\quad T' + \lambda T = 0$.

Boundary conditions: $u_x(0,t) = 0$, $u_x(\pi,t) = 0$

$\Rightarrow \left.\begin{aligned} X'(0)T(t) &= 0 \\ X'(\pi)T(t) &= 0 \end{aligned}\right\}$ either $T(t) = 0$ for all t (trivial) or $X'(0) = X'(\pi) = 0$

So we get the 2 point boundary value problem

$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(\pi) = 0 \end{cases}$ $\quad r^2 + \lambda = 0$

Look at the cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$ separately.

(1.) $\lambda > 0 \Rightarrow r_{1,2} = \mp i\sqrt{\lambda}$

$X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$X'(x) = -\sqrt{\lambda} c_1 \sin(\sqrt{\lambda} x) + \sqrt{\lambda} c_2 \cos(\sqrt{\lambda} x)$. $\quad X'(0) = 0 \Rightarrow c_2 = 0$

$0 = X'(\pi) = -\sqrt{\lambda} c_1 \sin(\sqrt{\lambda}\pi) \rightsquigarrow$ if $c_1 = 0$, then soln. is trivial.

So, $\sin(\sqrt{\lambda}\pi) = 0 \Rightarrow \lambda = n^2$, $\quad n = 1, 2, 3, \ldots$

$X_n(x) = \cos(nx)$

(2) $\lambda = 0$ $\quad r_1 = r_2 = 0$

$X(x) = c_1 + c_2 x$. $\quad X'(x) = c_2$

$X'(0) = X'(\pi) = 0 \iff c_2 = 0$

So $X_0(x) = 1$ is a nontrivial solution.

(3.) $\lambda < 0$ $\quad r_{1,2} = \mp\sqrt{-\lambda}$

$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \Rightarrow X'(x) = \sqrt{-\lambda} c_1 e^{\sqrt{-\lambda}x} - \sqrt{-\lambda} c_2 e^{-\sqrt{-\lambda}x}$

$\left.\begin{aligned} X'(0) &= 0 = c_1 - c_2 \\ X'(\pi) &= 0 = c_1 e^{\sqrt{-\lambda}\pi} - c_2 e^{-\sqrt{-\lambda}\pi} \end{aligned}\right\}$ $c_1 = c_2 = 0$ ~~all~~ solutions in this case are trivial.

For $\lambda = n^2$; $T' + n^2 T = 0 \Rightarrow T(t) = c e^{-n^2 t}$

$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} a_n \cdot \cos(nx) \cdot e^{-n^2 t}$$

$u(x,0) = 1 - \cos(4x)$ implies $a_0 = 1$, $a_4 = -1$ and all others are 0

$$\Rightarrow \boxed{u(x,t) = 1 - e^{-16t} \cdot \cos(4x)}$$