M E T U Department of Mathematics

Math 219 Introduction to Differential Equations Fall 2019 Final 5 January 2020 17:00		
F U L L N A M E	S T U D E N T I D	DURATION 140 MINUTES
6 QUESTIONS ON 4 PAGES	SHOW ALL YOUR WORK	TOTAL 100 POINTS

(20 pts) **1.** Find the general solution of the nonhomogeneous system

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 2 & 0 \\ 1 & 9 \end{bmatrix}}_{A} \mathbf{x} + \underbrace{e^{5t}\begin{bmatrix} 2 \\ 4 \end{bmatrix}}_{\vec{b}(t)}$$

by using **variation of parameters.**

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 9-\lambda \end{vmatrix} = (2-\lambda)(9-\lambda) = 0 \iff \boxed{\lambda = 2 \text{ or } \lambda = 9}$$

Eigenvectors for $\lambda = 2$: Solve $(A-2I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 7 & 0 \end{array}\right] \rightsquigarrow \vec{v} = \begin{bmatrix} -7 \\ 1 \end{bmatrix} k, \quad k \neq 0$$

Eigenvectors for $\lambda = 9$: Solve $(A-9I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} -7 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \rightsquigarrow \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot k, \quad k \neq 0.$$

We can form two independent homogeneous solutions

$\vec{x}^{(1)} = e^{2t}\begin{bmatrix} -7 \\ 1 \end{bmatrix}$, $\vec{x}^{(2)} = e^{9t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and a fundamental matrix

$$\Psi = \left[\vec{x}^{(1)} \middle| \vec{x}^{(2)}\right] = \begin{bmatrix} -7e^{2t} & 0 \\ e^{2t} & e^{9t} \end{bmatrix} \qquad \det \Psi = -7e^{11t}$$

$$\Psi^{-1} = \frac{1}{-7e^{11t}} \begin{bmatrix} e^{9t} & 0 \\ -e^{2t} & -7e^{2t} \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -e^{-2t} & 0 \\ e^{-9t} & 7e^{-9t} \end{bmatrix}$$

$$\boxed{\vec{x} = \Psi \int \Psi^{-1}\vec{b}\, dt} \qquad \Psi^{-1}\vec{b} = \frac{1}{7}\begin{bmatrix} -e^{-2t} & 0 \\ e^{-9t} & 7e^{-9t} \end{bmatrix}\begin{bmatrix} 2e^{5t} \\ 4e^{5t} \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -2e^{3t} \\ 30e^{-4t} \end{bmatrix}$$

$$\int \Psi^{-1}\vec{b}\, dt = \frac{1}{7}\begin{bmatrix} \int -2e^{3t}\, dt \\ \int 30e^{-4t}\, dt \end{bmatrix} = \begin{bmatrix} -\frac{2}{21}e^{3t} \\ -\frac{30}{28}e^{-4t} \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\vec{x} = \Psi \int \Psi^{-1}\vec{b}\, dt = \begin{bmatrix} -7e^{2t} & 0 \\ e^{2t} & e^{9t} \end{bmatrix}\left(\begin{bmatrix} -\frac{2}{21}e^{3t} \\ -\frac{30}{28}e^{-4t} \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right) = \begin{bmatrix} 2/3 \\ -7/6 \end{bmatrix}e^{5t} + c_1\begin{bmatrix} -7e^{2t} \\ e^{2t} \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{9t} \end{bmatrix}$$

$c_1, c_2 \in \mathbb{R}$

(20 pts) **2.** Solve the following initial value problem by using the Laplace transform:

$$y'' + 4y' + 4y = 2\delta(t-5), \quad y(0) = y'(0) = 1.$$

Apply Laplace transformation to both sides. Say $\mathcal{L}\{y\} = Y(s)$.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 2\mathcal{L}\{\delta(t-5)\}$$

$$s^2Y(s) - s - 1 + 4(sY(s) - 1) + 4Y(s) = 2\cdot e^{-5s}$$

$$(s^2+4s+4)Y(s) = s+5+2e^{-5s} \Rightarrow \boxed{Y(s) = \frac{s+5+2e^{-5s}}{\underbrace{s^2+4s+4}_{(s+2)^2}}}$$

$$Y(s) = \frac{s+2}{(s+2)^2} + \frac{3}{(s+2)^2} + \frac{2e^{-5s}}{(s+2)^2}$$

$$= \frac{1}{s+2} + \frac{3}{(s+2)^2} + \frac{2e^{-5s}}{(s+2)^2}$$

$$\boxed{y(t) = e^{-2t} + 3\cdot t\cdot e^{-2t} + 2u_5(t)\cdot(t-5)e^{-2(t-5)}}$$

(15 pts) **3.** Solve the initial value problem $y'' + 4y = g(t)$, $y(0) = y'(0) = 0$ by using the Laplace transform, where

$$g(t) = \begin{cases} \sin(t), & 0 \le t < 2\pi, \\ 0, & \text{otherwise} \end{cases}.$$

$$g(t) = \sin(t) + u_{2\pi}(t)\cdot(0 - \sin(t))$$
$$= \sin(t) + u_{2\pi}(t)\cdot(-\sin(t-2\pi))$$

Apply Laplace transfomation to both sides. Say $\mathcal{L}\{y\} = Y(s)$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$s^2Y(s) + 4Y(s) = \mathcal{L}\{\sin(t)\} + \mathcal{L}\{u_{2\pi}(t)\cdot(-\sin(t-2\pi))\}$$

$$(s^2+4)Y(s) = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1} \Rightarrow \boxed{Y(s) = (1-e^{-2\pi s})\cdot\frac{1}{(s^2+4)(s^2+1)}}$$

$$\frac{1}{(s^2+4)(s^2+1)} = \frac{A}{s^2+4} + \frac{B}{s^2+1}$$ (s terms not needed in numerator since everything is in terms of s^2)

$$1 = A(s^2+1) + B(s^2+4) \Rightarrow \left.\begin{array}{l} A+B=0 \\ A+4B=1 \end{array}\right\} \boxed{\begin{array}{l} A=-1/3 \\ B=1/3 \end{array}}$$

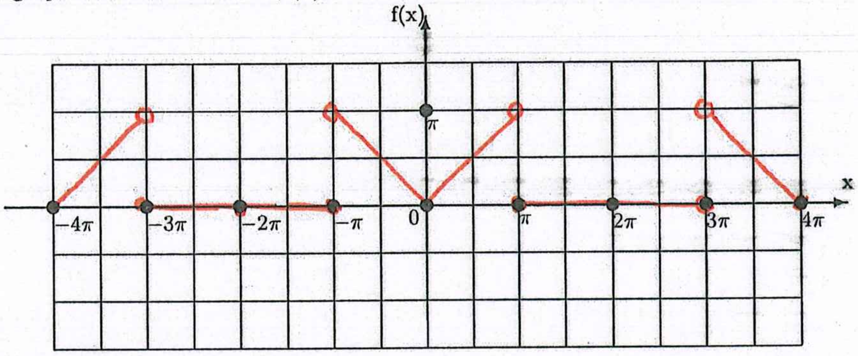
$$Y(s) = (1-e^{-2\pi s})\left(\frac{-1/3}{s^2+4} + \frac{1/3}{s^2+1}\right)$$

$$\boxed{y(t) = \frac{-1}{6}\sin(2t) + \frac{1}{3}\sin(t) - u_{2\pi}(t)\cdot\left(-\frac{1}{6}\sin(2(t-2\pi)) + \frac{1}{3}\sin(t-2\pi)\right)}$$

(15 pts) **4.** Suppose that the function f is defined on the interval $[0, 2\pi]$ by

$$f(x) = \begin{cases} x, & 0 \le x < \pi \\ 0, & \pi \le x \le 2\pi \end{cases}$$

(a) Draw a graph of the extension of $f(x)$ to $\mathbb{R}$ as an **even** function with period 4π.



(b) Find the Fourier series of the extended function in part (a).

Since the extended function is even, we'll just have a cosine series.

$$a_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)\cdot\cos\left(\frac{nx}{2}\right)dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos\left(\frac{nx}{2}\right)dx = \frac{1}{\pi}\int_0^{\pi} \underbrace{x}_{u}\cdot\underbrace{\cos\left(\frac{nx}{2}\right)dx}_{dv}$$

$(n \geq 1)$ $\left(\begin{array}{l} du = dx \\ v = \frac{2}{n}\sin\left(\frac{nx}{2}\right)\end{array}\right)$

$$= \frac{1}{\pi}\left(x\cdot\frac{2}{n}\sin\left(\frac{nx}{2}\right)\Big|_0^{\pi} - \int_0^{\pi}\frac{2}{n}\sin\left(\frac{nx}{2}\right)dx\right)$$

$$= \frac{1}{\pi}\left(\frac{2\pi}{n}\sin\left(\frac{n\pi}{2}\right) + \left(\frac{2^2}{n^2}\cos\left(\frac{nx}{2}\right)\right)\Big|_0^{\pi}\right)$$

$$\boxed{= \frac{2}{n}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{\pi n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{\pi n^2}}$$

$$a_0 = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx \boxed{= \frac{\pi}{2}}$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{nx}{2}\right)}$$

(10 pts) **5.** Consider the first order ODE $y' = y(6-y)g(y)$ where g is a polynomial. Prove that if $y(0) = 4$, then $0 < y(t) < 6$ for all t in the domain of the solution of this initial value problem (Hint: The Existence-Uniqueness theorem may be useful for this proof).

Say $f(t,y) = y\cdot(6-y)\cdot g(y)$.
Both f and $\frac{\partial f}{\partial y}$ are continuous on all of $\mathbb{R}^2$, so for any initial condition $y(t_0) = y_0$, there is a unique solution of the ODE satisfying this condition.
Observe that $y=0$ and $y=6$ are both solutions of the ODE.
If $y(0) = 4$ and $y(t) > 6$ for some t, then by intermediate value thm., there exists $t_0 \in (0,t)$ such that $y(t_0) = 6$.
But then there would be two solns. with $y(t_0) = 6$, contradiction.
Same if $y(t) < 0$ for some t.

(20 pts) **6.** Find the solution of the problem

$$u_t + tu_{xx} = 0, \quad 0 < x < 10,\ 0 < t$$
$$u(0,t) = u(10,t) = 0, \quad 0 < t$$
$$u(x,0) = \sin\left(\frac{\pi x}{10}\right), \quad 0 < x < 10.$$

Show all steps of your work, including the detailed analysis of the two point boundary value problem obtained.

Say $u(x,t) = X(x)\cdot T(t)$. Then $u_t + tu_{xx} = 0$ implies

$X \cdot T' + t X'' \cdot T = 0$

$\Rightarrow \boxed{\underbrace{\frac{X''}{X}}_{\text{must be constant}} = \frac{-T'}{t\cdot T} = -\lambda}$ where λ is a constant

$u(0,t) = u(10,t) = 0 \Rightarrow$ either $T(t) = 0$ for all t (trivial)

or $\boxed{X(0) = X(10) = 0}$

$\left.\begin{array}{l} X'' + \lambda X = 0 \\ X(0) = X(10) = 0 \end{array}\right\}$ $r^2 + \lambda = 0$. Look at the cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

(i) $\lambda > 0$: $r_{1,2} = \mp i\sqrt{\lambda} \Rightarrow X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

$X(0) = 0 \Rightarrow c_1 = 0$.

$X(10) = 0 \Rightarrow$ either $c_2 = 0$ (trivial) or $\boxed{\sin(10\sqrt{\lambda}) = 0}$

$\boxed{\lambda = \frac{n^2\pi^2}{100}} \rightsquigarrow$ get $\boxed{X_n(x) = \sin\left(\frac{n\pi x}{10}\right)}$

$n = 1, 2, 3, \ldots$

(ii) $\lambda = 0$ $r_1 = r_2 = 0 \Rightarrow X(x) = c_1 + c_2 x$.

$X(0) = X(10) = 0$ implies $c_1 = c_2 = 0$.

(iii) $\lambda < 0$: $r_{1,2} = \mp\sqrt{-\lambda} \Rightarrow X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$

$\left.\begin{array}{l} X(0) = 0 \Rightarrow c_1 + c_2 = 0 \\ X(10) = 0 \Rightarrow c_1 e^{10\sqrt{-\lambda}} + c_2 e^{-10\sqrt{-\lambda}} = 0 \end{array}\right\}$ $c_1 = c_2 = 0$

For $\lambda = \frac{n^2\pi^2}{100}$, $T' - \lambda t T = 0$ can be solved as follows:

$\int \frac{T'}{T} dt = \int \lambda t\, dt \Rightarrow \ln|T| = \frac{\lambda t^2}{2} + c \Rightarrow T(t) = ce^{\frac{\lambda t^2}{2}}$

Set $T_n(t) = e^{\frac{n^2\pi^2 t^2}{200}}$, $u_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi x}{10}\right)\cdot e^{\frac{n^2\pi^2 t^2}{200}}$

$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{10}\right) e^{\frac{n^2\pi^2 t^2}{200}}$. To meet the condition

$u(x,0) = \sin\left(\frac{\pi x}{10}\right)$, set $c_1 = 1$, $c_n = 0, n \neq 1$. So, $\boxed{u(x,t) = \sin\left(\frac{\pi x}{10}\right) e^{\frac{\pi^2 t^2}{200}}}$