M E T U
Department of Mathematics

	Introduction to Differential Equations MidTerm 2	

Code: *Math 219* Semester: *Spring 2019* Date: *4 May 2019* Time: *17:00* Duration: *120 minutes*	Last Name: Name: Department: Student No.: Signature: 6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	SHOW YOUR WORK	

Question 1 (20 pts) Solve the initial value problem

$$y'' - y = te^t - 2e^{-t} + 1, \quad y(0) = 0, y'(0) = \frac{23}{4}.$$

y_h: $y'' - y = 0 \rightsquigarrow \lambda^2 - 1 = 0$, $\lambda_1 = +1$, $\lambda_2 = -1$

$y_h = c_1 e^t + c_2 e^{-t}$

y_p: Try $y_p = At^2e^t + Bte^t + Cte^{-t} + D$

$y_p' = At^2e^t + (2A+B)te^t + Be^t - Cte^{-t} + Ce^{-t}$

$y_p'' = At^2e^t + (4A+B)te^t + (2A+2B)e^t + Cte^{-t} - 2Ce^{-t}$

$y_p'' - y_p = 4Ate^t + (2A+2B)e^t - 2Ce^{-t} - D = te^t - 2e^{-t} + 1$

$$\left.\begin{array}{l} 4A = 1 \\ 2A + 2B = 0 \\ -2C = -2 \\ -D = 1 \end{array}\right\} \begin{array}{l} A = 1/4, \ B = -1/4 \\ C = 1, \ D = -1 \end{array}$$

$y_p = \frac{1}{4}t^2e^t - \frac{1}{4}te^t + te^{-t} - 1$

$y = y_h + y_p = c_1e^t + c_2e^{-t} + \frac{1}{4}t^2e^t - \frac{1}{4}te^t + te^{-t} + 1$

$y' = c_1e^t - c_2e^{-t} + \frac{1}{4}t^2e^t + \frac{1}{4}te^t - \frac{1}{4}e^t - te^{-t} + e^{-t}$

$$\left.\begin{array}{l} y(0) = 0 = c_1 + c_2 + 1 \\ y'(0) = \frac{23}{4} = c_1 - c_2 - \frac{1}{4} + 1 \end{array}\right\} c_1 - c_2 = 5 \ \left.\right\} \begin{array}{l} c_1 = 2 \\ c_2 = -3 \end{array}$$

$$\boxed{y = 2e^t - 3e^{-t} + \frac{1}{4}t^2e^t - \frac{1}{4}te^t + te^{-t} + 1}$$

Question 2 (15 pts) Suppose that an object with mass $4\,kg$ is attached to a linear spring with spring constant $40\,kg/s^2$. Assume that the damping coefficient is $8\,kg/s$ and a constant external force of $F_0\,kg \cdot m/s^2$ is applied to the object.

(a) Write a differential equation that models this system and find its solutions. Is the system underdamped, critically damped or overdamped?

$$4u'' + 8u' + 40u = F_0$$

u_h: $4u'' + 8u' + 40u = 0 \rightsquigarrow 4\lambda^2 + 8\lambda + 40 = 0$

$$\lambda^2 + 2\lambda + 10 = 0$$

$$(\lambda+1)^2 + 9 = 0 \Rightarrow \lambda_{1,2} = -1 \mp 3i$$

$u_h(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)$ System is underdamped since it has non-real roots.

u_p: Try $y_p = A$. $y_p' = y_p'' = 0$

$40A = F_0 \Rightarrow A = \frac{F_0}{40}$

$$\boxed{\begin{aligned} u(t) &= u_h(t) + u_p(t) \\ &= c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \frac{F_0}{40} \end{aligned}}$$

(b) It is observed that as time increases, the position of the object approaches a steady state which is $3\,m$ below the equilibrium point. Determine F_0.

It is given that $\lim_{t\to+\infty} u(t) = 3$ m.

On the other hand

$$\lim_{t\to+\infty} u(t) = \lim_{t\to+\infty}\left(c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \frac{F_0}{40}\right) = \frac{F_0}{40}$$

$$\frac{F_0}{40} = 3 \Rightarrow \boxed{F_0 = 120}$$

Question 3 (15 pts) Find a non-homogeneous, constant coefficient, linear ODE of the lowest possible order, such that $y_1 = \cos(2t) + e^{-t}$, $y_2 = e^{-t} + e^{2t}$ are two of its solutions.

$y_1 - y_2 = \cos(2t) + \cancel{e^{-t}} - \cancel{e^{-t}} - e^{2t} = \cos(2t) - e^{2t}$

must be a solution of the associated homogeneous equation.

Since $\cos(2t)$ corresponds to the roots $\mp 2i$ and e^{2t} corresponds to the root 2, the characteristic polynomial of the eqn. must be divisible by

$(\lambda - 2i)(\lambda + 2i)(\lambda - 2) = (\lambda^2 + 4)(\lambda - 2) = \lambda^3 - 2\lambda^2 + 4\lambda - 8$.

So the eqn. must at least be of 3^{rd} order, and for minimal order we can assume char. poly. is exactly $\lambda^3 - 2\lambda^2 + 4\lambda - 8$.

$\Rightarrow$ The eqn. is $y''' - 2y'' + 4y' - 8y = g(t)$ for some $g(t)$

$y_h = c_1\cos(2t) + c_2\sin(2t) + c_3 e^{2t}$. We observe that $y_p = e^{-t}$ is a particular soln. $g(t) = y_p''' - 2y_p'' + 4y_p' - 8y_p = (-1 - 2 - 4 - 8)e^{-t}$

$$\Rightarrow \boxed{y''' - 2y'' + 4y' - 8y = -15e^{-t}}$$

Question 4 (20 pts) Consider the initial value problem (IVP)

$$(x+1)(x-2)y''+6xy'+3y=0, \quad y(0)=-4,\ y'(0)=2.$$

(a) Show that $x_0=0$ is an ordinary point.

Say $P(x)=(x+1)(x-2)$, $Q(x)=6x$, $R(x)=3$. Then

$\frac{Q(x)}{P(x)}=\frac{6x}{(x+1)(x-2)}$ and $\frac{R(x)}{P(x)}=\frac{3}{(x+1)(x-2)}$ are both analytic in a neighborhood of $x_0=0$, since they are both ratios of polynomials with nonvanishing denominators at $x=0$. So, $x_0=0$ is ordinary.

(b) For the power series solution of the IVP centered at $x_0=0$, find the recursion relation and compute the first 5 non-zero terms.

$$y=\sum_{n=0}^{\infty}a_nx^n, \quad y'=\sum_{n=1}^{\infty}na_nx^{n-1}, \quad y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$$

$$(x+1)(x-2)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+6x\cdot\sum_{n=1}^{\infty}na_nx^{n-1}+3\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=1}^{\infty}(n+1)n\,a_{n+1}x^n$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^n-\sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}-\sum_{n=2}^{\infty}2n(n-1)a_nx^{n-2}$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n$$

$$+\sum_{n=1}^{\infty}6na_nx^n+\sum_{n=0}^{\infty}3a_nx^n=0$$

$$(-2\cdot2\cdot1\cdot a_2+3a_0)x^0+(-2\cdot1\cdot a_2-2\cdot3\cdot2\cdot a_3+6\cdot a_1+3a_1)x^1$$

$$+\sum_{n=2}^{\infty}\Big[n(n-1)a_n-(n+1)n\,a_{n+1}-2(n+2)(n+1)a_{n+2}+6na_n+3a_n\Big)x^n=0$$

$\boxed{a_0=y(0)=-4 \quad a_1=y'(0)=2}$ $\quad\boxed{a_2=\frac{3}{4}a_0=-3}$ $\quad 12a_3=9a_1-2a_2 = 18+6=24 \Rightarrow \boxed{a_3=2}$

$n\geq2 \Rightarrow 2(n+2)(n+1)a_{n+2}=-n(n+1)a_{n+1}+(n^2+5n+3)a_n$

$$\boxed{a_{n+2}=\frac{-n(n+1)a_{n+1}+(n^2+5n+3)a_n}{2(n+1)(n+2)}, \quad n\geq2}$$

$a_4=\frac{-2\cdot3\cdot a_3+17a_2}{2\cdot3\cdot4}$ recursion

$\Rightarrow \boxed{a_4=-21/8}$ $\quad\boxed{y=-4+2x-3x^2+2x^3-\frac{21}{8}x^4+\cdots}$

(c) What can be said about the radius of convergence of the power series solution above?

The singular points are -1 and 2.

The radius of convergence ρ is $\geq$ the distance from x_0 to the nearest singular point.

Hence $\boxed{\rho\geq1}$

Question 5 (15 pts) Find all solutions of the Cauchy-Euler equation

$$x^2y'' - xy' + y = 0, \quad x > 0.$$

Let $x=e^t$. Then $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}=\frac{dy}{dx}.x$

$\frac{d^2y}{dt^2}=\frac{d}{dt}\left(\frac{dy}{dt}\right)=\frac{d}{dt}\left(\frac{dy}{dx}.x\right)=\frac{d}{dx}\left(\frac{dy}{dx}.x\right)\frac{dx}{dt}=\frac{d^2y}{dx^2}x^2+\frac{dy}{dx}x$

So; $\frac{d^2y}{dt^2}-2\frac{dy}{dt}+y=0 \Rightarrow r^2-2r+1=0 \Rightarrow r_1=r_2=1$

$y(t)=c_1.e^t+c_2.t.e^t \Rightarrow \boxed{y(x)=c_1x+c_2x\ln(x)}$ where $c_1,c_2\in\mathbb{R}$

Question 6 (15 pts) Consider the ODE $(1-x^2)y''-xy'+k^2y=0$ where k is a constant.

$P(x)=1-x^2 \quad Q(x)=-x \quad R(x)=k^2$

(a) Find all ordinary points and singular points. Determine if the singular points are regular or not.

$\frac{Q(x)}{P(x)}=\frac{-x}{1-x^2}$, $\frac{R(x)}{P(x)}=\frac{k^2}{1-x^2}$ are analytic except ∓1, So ∓1 are singular points. any other points is ordinary point.

Regularity of +1: $(x-1).\frac{Q(x)}{P(x)}=\frac{x}{1+x}$, $(x-1)^2\frac{R(x)}{P(x)}=\frac{(1-x)k^2}{1+x}$ are both analytic near 1.

So, +1 is a regular singular point.

Regularity of −1: $(x+1)\frac{Q(x)}{P(x)}=\frac{x}{x-1}$, $(x+1)^2\frac{R(x)}{P(x)}=\frac{(x+1)k^2}{1-x}$ are both analytic near −1

So, −1 is a regular singular point.

(b) Write the form of two linearly independent series solutions with center $x_0=1$. (Do not compute the coefficients.)

$\alpha=\lim_{x\to1}(x-1)\frac{Q(x)}{P(x)}=\lim_{x\to1}\frac{x}{1+x}=\frac{1}{2}$, $\beta=\lim_{x\to1}(x-1)^2\frac{R(x)}{P(x)}=\lim_{x\to1}\frac{(1-x)k^2}{1+x}=0$

So the associated Cauchy-Euler equation is $(x-1)^2y''+\frac{1}{2}(x-1)y'+0.y=0$

Indicial eqn: $r^2+(\alpha-1)r+\beta=0 \Rightarrow r^2-\frac{1}{2}r=0 \Rightarrow r_1=0,\ r_2=\frac{1}{2}$

Solutions of C-E eqn: $c_1|x-1|^{1/2}+c_2$. r_1-r_2 is not an integer.

So two independent solutions will have the form:

$\boxed{y_1(x)=|x-1|^{1/2}\sum_{n=0}^{\infty}a_n(x-1)^n}$, $\boxed{y_2(x)=\sum_{n=0}^{\infty}b_n(x-1)^n}$