M E T U Department of Mathematics

Math 219 Introduction to Differential Equations Fall 2019 Midterm 2 21 December 2019 13:30		
FULL NAME	STUDENT ID	DURATION 120 MINUTES
5 QUESTIONS ON 4 PAGES	SHOW ALL YOUR WORK	TOTAL 120 POINTS

(30 pts) **1.** Find the general solution of the following differential equation, by using the method of undetermined coefficients:

$$y''' - 4y'' + 5y' - 2y = t(e^t + 1).$$

Observe that $\boxed{r=1}$ is a root of the characteristic equation $r^3 - 4r^2 + 5r - 2 = 0$. To find the other roots, do polynomial division.

$$\begin{array}{l|l} r^3-4r^2+5r-2 & r-1 \\ \underline{-\;\; r^3-r^2} & r^2-3r+2 \\ \quad -3r^2+5r-2 & \\ \underline{-\;\; -3r^2+3r} & \\ \qquad 2r-2 & \end{array}$$

$$r^3-4r^2+5r-2 = (r-1)(r^2-3r+2) = (r-1)^2(r-2)$$

So, $\boxed{r_1 = r_2 = 1 \, , \; r_3 = 2}$

Consequently, $\boxed{y_h = c_1 e^t + c_2 te^t + c_3 e^{2t}}$

Try $\boxed{y_p = t^2(Ate^t + Be^t) + (Ct + D)}$

since the root $r=1$ appears as a double root for y_h

$$y_p' = At^3e^t + (3A+B)t^2e^t + (2B)te^t + C$$

$$y_p'' = At^3e^t + (6A+B)t^2e^t + (6A+4B)te^t + (2B)e^t$$

$$y_p''' = At^3e^t + (9A+B)t^2e^t + (18A+6B)te^t + (6A+6B)e^t$$

$$y_p''' - 4y_p'' + 5y_p' - 2y_p = (-6A)te^t + (6A-2B)e^t + (-2C)t + (5C-2D)$$
$$= te^t + t$$

$$\left.\begin{array}{l} -6A = 1 \\ 6A - 2B = 0 \end{array}\right\} \boxed{A = -1/6 \, , \; B = -1/2} \qquad \left.\begin{array}{l} -2C = 1 \\ 5C - 2D = 0 \end{array}\right\} \boxed{\begin{array}{l} C = -1/2 \\ D = -5/4 \end{array}}$$

$$y_p = \frac{-1}{6}t^3e^t - \frac{1}{2}t^2e^t - \frac{1}{2}t - \frac{5}{4}$$

$$\Rightarrow \boxed{y = y_h + y_p = c_1e^t + c_2te^t + c_3e^{2t} - \frac{1}{6}t^3e^t - \frac{1}{2}t^2e^t - \frac{1}{2}t - \frac{5}{4}}$$

(30 pts) **2.** Consider the differential equation

$$(3-x^2)y'' - 3xy' - y = 0.$$

(a) Show that $x_0 = 0$ is an ordinary point.

$\frac{-3x}{3-x^2}$ and $\frac{-1}{3-x^2}$ are rational functions and continuous at 0, hence they are analytic. Therefore $x_0 = 0$ is an ordinary point.

(b) Find the recurrence relation for a power series solution centered at $x_0 = 0$.

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$0 = (3-x^2)y'' - 3xy' - y = 3\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 3\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= 3\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 3\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= (3\cdot 2\cdot 1\cdot a_2 - a_0)x^0 + (3\cdot 3\cdot 2\cdot a_3 - 3\cdot 1\cdot a_1 - a_1)x^1 + \sum_{n=2}^{\infty}\left[3(n+2)(n+1)a_{n+2} - n(n-1)a_n - 3na_n - a_n\right]x^n$$

Equating the coefficients of each x^n to 0, we get

$\boxed{6a_2 - a_0 = 0}$, $\quad 3(n+2)(n+1)a_{n+2} - \underbrace{(n^2+2n+1)}_{(n+1)^2}a_n = 0 \quad (n \geq 2)$

$\boxed{18a_3 - 4a_1 = 0}$,

$$\boxed{a_{n+2} = \frac{n+1}{3(n+2)} a_n \quad (n \geq 2)}$$

(recursion relation)

(c) Find the first 5 nonzero terms of a power series solution centered at $x_0 = 0$.

$a_2 = a_0/6, \quad a_3 = 2a_1/9, \quad a_4 = a_2/4 = a_0/24$

$$\boxed{\begin{aligned} y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ &= a_0 + a_1 x + \frac{a_0}{6}x^2 + \frac{2a_1}{9}x^3 + \frac{a_0}{24}x^4 + \cdots \end{aligned}}$$

(any nonzero values of a_0, a_1 would give a solution of the desired form).

(20 pts) **3.** Consider the differential equation

$$(x^3-x^2)y''+2(e^x-1)y'+\frac{\cos(x)}{(x-1)^2}y=0.$$

(a) Find and classify all singular points.

$\frac{2(e^x-1)}{x^3-x^2}$ and $\frac{\cos(x)}{(x-1)^2(x^3-x^2)}$ are analytic except $x=0$ and 1

So 0 and 1 are the singular points.

$x=1$ is not regular: $(x-1)^2\cdot\frac{\cos(x)}{(x-1)^2(x^3-x^2)}$ is not analytic at 1.

$x=0$ is regular: $\frac{x\cdot 2(e^x-1)}{(x^3-x^2)}=\frac{2(e^x-1)}{x(x-1)}=\frac{2\left(1+x+\frac{x^2}{2}+\cdots-1\right)}{x(x-1)}=\frac{2\left(1+\frac{x}{2}+\cdots\right)}{x-1}$

is analytic at 0. ~~Al~~Also, $\frac{x^2\cos(x)}{(x^3-x^2)(x-1)^2}=\frac{\cos(x)}{(x-1)^3}$ is analytic at 0.

(b) For one of the regular singular points, find the roots of the indicial equation and write the form of the series solutions around that point. Look at $x_0=0$.

$\alpha=\lim_{x\to0}\frac{x\cdot2(e^x-1)}{(x^3-x^2)}=\lim_{x\to0}\frac{2(1+\frac{x}{2}+\cdots)}{x-1}=-2$, $\beta=\lim_{x\to0}\frac{x^2\cos(x)}{(x^3-x^2)(x-1)^2}=-1$

$r^2+(\alpha-1)r+\beta=0 \rightsquigarrow \boxed{r^2-3r-1=0}$ $\quad r_{1,2}=\frac{3\mp\sqrt{9+4}}{2}=\frac{3\mp\sqrt{13}}{2}$

indicial equation

$r_2-r_1=\sqrt{13}$ is not an integer. So, $y=y_1+y_2$ where

$$\boxed{y_1=|x|^{\frac{3+\sqrt{13}}{2}}\cdot\sum_{n=0}^{\infty}a_nx^n}\ ,\quad \boxed{y_2=|x|^{\frac{3-\sqrt{13}}{2}}\sum_{n=0}^{\infty}b_nx^n}$$

(20 pts) **4.** Solve the differential equation $y''+y=\underbrace{\frac{1}{\cos(x)}}_{g(x)}$, $|x|<\frac{\pi}{2}$ by using variation of parameters:

$r^2+1=0\Rightarrow r_{1,2}=\mp i \Rightarrow \boxed{y_h=c_1\underbrace{\cos(x)}_{y_1}+c_2\underbrace{\sin(x)}_{y_2}}$

$$W(y_1,y_2)=\begin{vmatrix}\cos(x) & \sin(x)\\ -\sin(x) & \cos(x)\end{vmatrix}=\cos^2(x)+\sin^2(x)=1$$

$$y=y_1\int\frac{-y_2\cdot g(x)}{W}dx+y_2\int\frac{y_1\cdot g(x)}{W}dx$$

$$=\cos(x)\int\frac{-\sin(x)}{\cos(x)}dx+\sin(x)\int\frac{\cos(x)}{\cos(x)}dx$$

$$=\cos(x)\left(\ln|\cos(x)|+c_1\right)+\sin(x)\cdot\left(x+c_2\right)$$

($\cos(x)>0$ in the given domain)

$$\boxed{=\underbrace{c_1\cdot\cos(x)+c_2\cdot\sin(x)}_{y_h}+\underbrace{\cos(x)\cdot\ln(\cos(x))+x\cdot\sin(x)}_{y_p}}$$

(Bonus question: 20 pts) **5.** Suppose that $\omega > 0$ and $\omega \neq 2$. Consider the differential equation

$$y'' + 4y = 10\cos(\omega t).$$

(a) Show that for any value of ω, each solution of this differential equation is bounded.

$r^2+4=0 \Rightarrow r_{1,2} = \mp 2i \Rightarrow \boxed{y_h = c_1 \cdot \cos(2t) + c_2 \cdot \sin(2t)}$

Try $y_p = A\cos(\omega t) + B\sin(\omega t)$

$y_p' = -\omega A \sin(\omega t) + \omega B\cos(\omega t)$

$y_p'' = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$

$$y_p'' + 4y_p = (-\omega^2 A + 4A)\cos(\omega t) + (-\omega^2 B + 4B)\sin(\omega t) = 10 \cdot \cos(\omega t)$$

$\Rightarrow \left.\begin{array}{l}(-\omega^2+4)A = 10 \\ (-\omega^2+4)B = 0\end{array}\right\} A = \dfrac{10}{4-\omega^2}, \quad B = 0$

$y_p = \dfrac{10}{4-\omega^2}\cos(\omega t)$.

$$\boxed{y = y_h + y_p = c_1 \cdot \cos(2t) + c_2 \cdot \sin(2t) + \frac{10}{4-\omega^2} \cdot \cos(\omega t)}$$

For any such solution,

$$|y| \leqslant |c_1| \cdot \underbrace{|\cos(2t)|}_{\leq 1} + |c_2| \cdot \underbrace{|\sin(2t)|}_{\leq 1} + \left|\frac{10}{4-\omega^2}\right| \cdot \underbrace{|\cos(\omega t)|}_{\leq 1}$$

$$\leqslant |c_1| + |c_2| + \left|\frac{10}{4-\omega^2}\right|,$$ hence it is bounded.

(b) Suppose that $y(0) = y'(0) = 0$ and $\omega \geq 5$. Show that $|y(t)| \leq 1$ for all t.

$y(0) = 0 = c_1 + \dfrac{10}{4-\omega^2} \Rightarrow \boxed{c_1 = \dfrac{-10}{4-\omega^2}}$

$y'(t) = -2c_1 \sin(2t) + 2c_2\cos(2t) - \dfrac{10\omega}{4-\omega^2}\sin(\omega t)$

$y'(0) = 0 = 2c_2 \Rightarrow \boxed{c_2 = 0}$

$$y(t) = \frac{-10}{4-\omega^2} \cdot \cos(2t) + \frac{10}{4-\omega^2} \cdot \cos(\omega t)$$

$$|y(t)| \leqslant \left|\frac{10}{\omega^2-4}\right| \cdot \underbrace{|\cos(2t)|}_{\leq 1} + \left|\frac{10}{\omega^2-4}\right| \cdot \underbrace{|\cos(\omega t)|}_{\leq 1}$$

$$\leqslant \left|\frac{20}{\omega^2-4}\right| \underset{\text{for } \omega \geqslant 5}{\leqslant} \frac{20}{25-4} < 1$$