M E T U
Department of Mathematics

	Introduction to Differential Equations MidTerm 2	

Code: *Math 219* Semester: *Fall 2018* Date: *22 December 2018* Time: *13:30* Duration: *120 minutes*	Last Name: Name: Department: Student No.: Signature: 5 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	SHOW YOUR WORK	

Question 1 (20 pts) Consider the differential equation

$$x^2y'' - 6y = \ln(x), \quad x > 0.$$

(a) Verify that $y_1 = x^3$ and $y_2 = x^{-2}$ are solutions of the equation $x^2y'' - 6y = 0, \quad x > 0.$

$$x^2(x^3)'' - 6x^3 = x^2 \cdot 6x - 6x^3 = 0$$

$$x^2(x^{-2})'' - 6x^{-2} = x^2 \cdot (-2)(-3)x^{-4} - 6x^{-2} = 0$$

(b) Use the method of variation of parameters to find the general solution of the non-homogeneous equation above.

$$W(x^3, x^{-2}) = \begin{vmatrix} x^3 & x^{-2} \\ 3x^2 & -2x^{-3} \end{vmatrix} = -5 \neq 0$$

So, $\{x^3, x^{-2}\}$ is linearly indep.

$$y'' - \frac{6}{x^2}y = \frac{\ln(x)}{x^2}$$

By variation of parameters;

$$y = x^3 \int \frac{-x^{-2} \cdot \frac{\ln(x)}{x^2}}{-5}\,dx + x^{-2} \int \frac{x^3 \cdot \frac{\ln(x)}{x^2}}{-5}\,dx$$

$$= +\frac{x^3}{5} \int \frac{\ln(x)}{x^4}\,dx - \frac{x^{-2}}{5} \int x \cdot \ln(x)\,dx$$

$$\int \frac{\ln(x)}{x^4}\,dx = \frac{x^{-3}}{-3} \cdot \ln(x) - \int \frac{x^{-3}}{-3} \frac{dx}{x} = \frac{-\ln(x)}{3x^3} + \frac{1}{3} \int \frac{dx}{x^4}$$

$u = \ln(x)$, $du = \frac{dx}{x}$; $dv = \frac{dx}{x^4}$, $v = \frac{x^{-3}}{-3}$

$$= -\frac{\ln(x)}{3x^3} - \frac{1}{9x^3} + c_1$$

$$\int x \cdot \ln(x)\,dx = \frac{x^2}{2} \cdot \ln(x) - \int \frac{x^2}{2} \frac{dx}{x} = \frac{x^2}{2} \cdot \ln(x) - \frac{1}{2} \int x\,dx$$

$u = \ln(x)$, $du = \frac{dx}{x}$; $dv = x\,dx$, $v = x^2/2$

$$= \frac{x^2}{2} \ln(x) - \frac{x^2}{4} + c_2$$

$$y = \frac{x^3}{5}\left(-\frac{\ln(x)}{3x^3} - \frac{1}{9x^3} + c_1\right) + \left(-\frac{1}{5x^2}\right)\left(\frac{x^2 \cdot \ln(x)}{2} - \frac{x^2}{4} + c_2\right)$$

$$\boxed{= -\frac{\ln(x)}{6} + \frac{1}{36} + c_1 x^3 + c_2 x^{-2} \quad \text{with } c_1, c_2 \in \mathbb{R}}$$

(new c_1, c_2).

Question 2 (25 pts) Consider an undamped spring-mass system where a 2 kg mass is attached to a linear spring with spring constant $k = 8\,\mathrm{kg}/s^2$. Parts (a) and (b) below are two independent questions about this system.

(a) Suppose that there is no external force on the object, which is initially pulled 3 m below the equilibrium position, and given an upward initial velocity so that the resulting motion has an amplitude of 6 m. Find the position $u(t)$ of the object at any time, and sketch its graph.

Say positive axis is downwards

$m = 2\,\mathrm{kg}$

$k = 8\,\mathrm{kg}/s^2$

$2u'' + 8u = 0$

$u'' + 4u = 0$

$\lambda^2 + 4 = 0$

$\lambda = \mp 2i$

$u(t) = c_1 \cos(2t) + c_2 \sin(2t)$

$u(0) = 3 \Rightarrow c_1 = 3$

Amplitude $= \sqrt{c_1^2 + c_2^2} = 6$

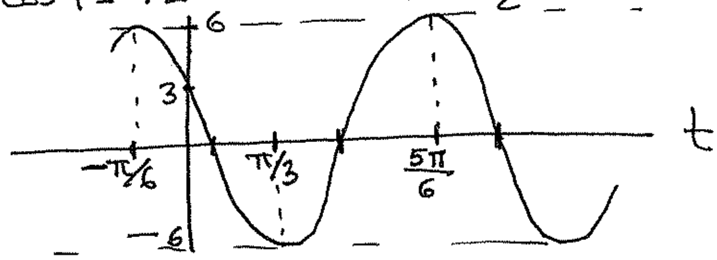
$\Rightarrow |c_2| = 3\sqrt{3}$

Since $u'(0)$ is upward $\Rightarrow$ negative, we have $c_2 = -3\sqrt{3}$

$u(t) = 3\cos(2t) - 3\sqrt{3}\sin(2t)$

$= 6\cos(2t - \varphi)$ where

$\cos\varphi = 1/2$ and $\sin\varphi = \frac{-\sqrt{3}}{2} \Rightarrow \varphi = -\pi/3$



(b) Suppose that the system is initially at rest. A downward external force $F(t) = 40\cos(\omega t)$ acts on the object, which causes $|u(t)|$ to be unbounded (causes resonance) as $t \to +\infty$. Find the position $u(t)$ of the object at any time.

$\omega = 2$ in order to have resonance.

$2u'' + 8u = 40\cos(2t)$

$u_h = c_1\cos(2t) + c_2\sin(2t)$

$u_p = At\cos(2t) + Bt\sin(2t)$

$u_p' = -2At\sin(2t) + A\cos(2t) + 2Bt\cos(2t) + B\sin(2t)$

$u_p'' = -4At\cos(2t) - 4A\sin(2t) - 4Bt\sin(2t) + 4B\cos(2t)$

$t\cos(2t)$ and $t\sin(2t)$ terms cancel in $2u_p'' + 8u_p$.

$-8A\sin(2t) + 8B\cos(2t) = 40\cos(2t)$

$\Rightarrow A = 0,\ B = 5$

$u_p = 5t\sin(2t)$

$u = u_h + u_p$

$= c_1\cos(2t) + c_2\sin(2t) + 5t\sin(2t)$

System is initially at rest $\Rightarrow u(0) = u'(0) = 0$.

$u(0) = c_1 = 0$

$u' = 2c_2\cos(2t) + 5\sin(2t) + 10t\cos(2t)$

$u'(0) = 2c_2 = 0$

$\Rightarrow \boxed{u(t) = 5t\sin(2t)}$

Question 3 (10 pts) Find an ordinary linear differential equation with constant coefficients whose general solution is

$$y = c_1 + c_2 t + c_3 e^{2t} + \ln(t), \quad t > 0, \quad \text{where } c_1, c_2, c_3 \in \mathbb{R}.$$

$y_h = c_1 + c_2 t + c_3 e^{2t}$, so the characteistic equation must have roots $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 2$. So it is $\lambda^2(\lambda - 2) = \lambda^3 - 2\lambda^2$. Hence, the left hand side of the equation is $y''' - 2y''$. To find the right hand side, put $y = \ln(t)$. $y' = \frac{1}{t}$, $y'' = \frac{-1}{t^2}$, $y''' = \frac{2}{t^3}$

$$\Rightarrow \boxed{y''' - 2y'' = \frac{2}{t^3} + \frac{2}{t^2}}$$

Question 4 (20 pts) Consider the initial value problem

$$(x^2+1)y'' + xy' + xy = 0, \quad y(0) = 0, y'(0) = 3.$$

(a) Show that $x_0 = 0$ is an ordinary point.

$P(x) = x^2+1$, $Q(x) = x$, $R(x) = x$

$\frac{Q(x)}{P(x)} = \frac{x}{x^2+1}$ and $\frac{R(x)}{P(x)} = \frac{x}{x^2+1}$ are ratios of polynomials, and they are continuous at $0 \Rightarrow$ they are analytic. So $x_0 = 0$ is an ordinary point

(b) Find an open interval centered at $x_0 = 0$ on which a power series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ is guaranteed to converge (explain your answer).

The singularities are at $x^2+1=0$, namely, $x = \mp i$
The distance in the complex plane from $x_0 = 0$ to the nearest singular point is then 1. So, the series has a radius of convergence ≥ 1. So it is guaranteed to converge on $(-1, 1)$.

(c) Find the solution of the initial value problem (find the recurrence relation for the coefficients and find the first three non-zero terms of the solution).

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \cdot \sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

x^0: $2 \cdot 1 \cdot a_2 = 0$
x^1: $3 \cdot 2 \cdot a_3 + 1 \cdot a_1 + a_0 = 0 \Rightarrow a_3 = \frac{-a_0 - a_1}{6}$

$n \geq 2 \Rightarrow n(n-1)a_n + (n+2)(n+1)a_{n+2} + n a_n + a_{n-1} = 0$

$$\boxed{a_{n+2} = \frac{-n^2 a_n - a_{n-1}}{(n+2)(n+1)}, \ n \geq 2}$$ (recurrence relation)

$a_0 = y(0) = 0$, $a_1 = y'(0) = 3$.
$a_2 = 0$, $a_3 = \frac{-a_0 - a_1}{6} = \frac{-1}{2}$

$a_4 = \frac{-4a_2 - a_1}{4 \cdot 3} = \frac{-1}{4}$

$$\Rightarrow \boxed{y = 3x - \frac{x^3}{2} - \frac{x^4}{4} + \cdots}$$

Question 5 (25 pts) Consider the differential equation

$$x^2y'' + \left(x^2 + \frac{1}{4}\right) y = 0, \quad x > 0.$$

(a) Show that $x_0 = 0$ is a regular singular point.

$P(x) = x^2$, $Q(x) = 0$, $R(x) = x^2 + \frac{1}{4}$

$\frac{R(x)}{P(x)}$ is not analytic at $0 \Rightarrow x_0 = 0$ is singular.

$\frac{x\,Q(x)}{P(x)} = 0$ and $\frac{x^2 R(x)}{P(x)} = x^2 + \frac{1}{4}$ are analytic at $x_0 = 0$, so x_0 is regular singular.

(b) Determine the indicial equation and show that its roots are equal. Find one (nontrivial) series solution of the equation (find the recurrence relation and the first three nonzero terms).

$\alpha = \lim_{x\to 0} \frac{x\,Q(x)}{P(x)} = 0$, $\beta = \lim_{x \to 0} \frac{x^2 R(x)}{P(x)} = \frac{1}{4}$

Hence the indicial equation $r^2 + (\alpha - 1) r + \beta = 0$ is

$$r^2 - r + \frac{1}{4} = 0$$

$$\left(r - \frac{1}{2}\right)^2 = 0 \Rightarrow r_1 = r_2 = \frac{1}{2} \text{ (roots are equal).}$$

Set $y = x^{1/2} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$

$$y' = \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) a_n x^{n-\frac{1}{2}}, \quad y'' = \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)\left(n - \frac{1}{2}\right) a_n x^{n-\frac{3}{2}}$$

$$x^2 \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right) a_n x^{n-3/2} + \left(x^2 + \frac{1}{4}\right) \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} = 0$$

$$\sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right) a_n x^{n+1/2} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+\frac{5}{2}}}_{\sum_{n=2}^{\infty} a_{n-2} x^{n+\frac{1}{2}}} + \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} = 0$$

$x^{1/2}$: $\frac{1}{2}\left(\frac{-1}{2}\right) a_0 + \frac{1}{4} a_0 = 0$ ($0 = 0$, no restiction on a_0)

$x^{3/2}$: $\frac{3}{2} \cdot \frac{1}{2} \cdot a_1 + \frac{1}{4} a_1 = 0 \Rightarrow a_1 = 0$

$x^{n+1/2}$, $n \geq 2$: $\left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right) a_n + a_{n-2} + \frac{1}{4} a_n = 0$

$$\boxed{a_n = \frac{-a_{n-2}}{n^2} \quad (n \geq 2)}$$ recurrence relation.

$a_2 = \frac{-a_0}{4}$, $a_3 = \frac{-a_1}{9} = 0$, $a_4 = \frac{-a_2}{16} = \frac{a_0}{64}$

$$\Rightarrow \boxed{y = x^{1/2}\left(a_0 - \frac{a_0 x^2}{4} + \frac{a_0 x^4}{64} + \cdots\right)}$$