M E T U
Department of Mathematics

	Introduction to Differential Equations MidTerm 1	

Code: *Math 219* Semester: *Fall 2018* Date: *17 November 2018* Time: *13:30* Duration: *120 minutes*	Last Name: Name: Department: Student No.: Signature: 6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	SHOW YOUR WORK	

Question 1 (20 pts) Consider the non-exact differential equation

$$\left(\frac{1}{xy} - 3x^2\right) dx + \frac{1}{y^2} dy = 0.$$

(a) Find an integrating factor of the form $\mu = x^n$.

Multiply by x^n: $\left(\frac{x^{n-1}}{y} - 3x^{n+2}\right) dx + \frac{x^n}{y^2} dy = 0$

One needs: $\frac{\partial}{\partial y}\left(\frac{x^{n-1}}{y} - 3x^{n+2}\right) = \frac{\partial}{\partial x}\left(\frac{x^n}{y^2}\right)$

$\frac{-x^{n-1}}{y^2} = \frac{n x^{n-1}}{y^2}$. Equality should hold for all x, y

so $\boxed{n=-1}$, $\boxed{\mu = x^{-1}}$

(b) Find the solution curves of the given differential equation in the form $\Phi(x, y) = c$.

$\left(\frac{1}{x^2 y} - 3x\right) dx + \frac{1}{xy^2} dy = 0$. We need to find $\Phi(x,y)$ such that $\frac{\partial \Phi}{\partial x} = \frac{1}{x^2 y} - 3x$, $\frac{\partial \Phi}{\partial y} = \frac{1}{xy^2}$

$$\Phi(x,y) = \frac{-1}{xy} + h(x)$$

$$\frac{\partial}{\partial x}\Phi(x,y) = \frac{1}{x^2 y} + h'(x) = \frac{1}{x^2 y} - 3x$$

$\Rightarrow h'(x) = -3x$. We can pick $h(x) = \frac{-3x^2}{2}$

$\Phi(x,y) = \frac{-1}{xy} - \frac{3x^2}{2}$. Solutions: $\boxed{\frac{-1}{xy} - \frac{3x^2}{2} = c}$

(c) For each of the following initial conditions, determine the interval in which the solution is valid: (i) $y(1) = 2$, (ii) $y(1) = -2$.

(i) $y(1) = 2 \Rightarrow \frac{-1}{1\cdot 2} - \frac{3\cdot 1^2}{2} = -2 = c$

So $\frac{-1}{xy} - \frac{3x^2}{2} = -2 \Rightarrow \frac{-1}{xy} = \frac{3x^2}{2} - 2 \Rightarrow y = \frac{-2}{x(3x^2-4)}$

Interval should contain the point 1

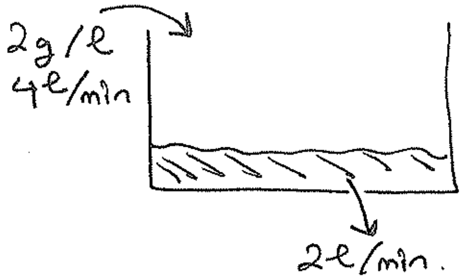
$\Rightarrow \boxed{\left(0, \frac{2}{\sqrt{3}}\right)}$

(ii) $y(1) = -2 \Rightarrow \frac{-1}{1\cdot(-2)} - \frac{3\cdot 1^2}{2} = -1 = c$

So $\frac{-1}{xy} - \frac{3x^2}{2} = -1 \Rightarrow \frac{-1}{xy} = \frac{3x^2}{2} - 1 \Rightarrow y = \frac{-2}{x(3x^2-2)}$

In this case, the relevant interval is $\boxed{\left(\frac{\sqrt{2}}{\sqrt{3}}, +\infty\right)}$

Question 2 (20 pts) A tank, with capacity 200ℓ (liters) contains 10ℓ of water in which is dissolved $40g$ (grams) of chemical. A solution containing $2g/\ell$ of the chemical flows into the tank at a rate of $4\ell/min$, and the well-stirred mixture flows out at a rate of $2\ell/min$. Determine the function $Q(t)$ describing the amount of the chemical in the tank at any time t before the tank overflows.



Volume at time t is $(10+2t)\ \ell$

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$
$$= 8\ g/min - \frac{Q}{10+2t} \cdot 2g/min$$

$$\frac{dQ}{dt} + \frac{Q}{5+t} = 8 \longrightarrow \text{First order, linear.}$$

$$\mu(t) = e^{\int \frac{1}{5+t}\,dt} = e^{\ln(5+t)} = 5+t$$

$$((5+t)\cdot Q)' = 8(5+t)$$

$$(5+t)\cdot Q = 40t + 4t^2 + c$$

$$Q(0) = 40g \Rightarrow 5\cdot 40 = c$$

$$\boxed{Q(t) = \frac{4t^2+40t+200}{5+t}\ g} \quad \text{for } t \leqslant 95 \text{ min}$$

Question 3 (10 pts) Suppose that $A(t)$ is a 2×2 matrix whose entries are continuous functions and $\mathbf{g}(t) \neq \mathbf{0}$ is a 2×1 vector of continuous functions. Assume that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ are solutions of the system $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{g}(t)$. Furthermore, suppose that

$$\mathbf{x}^{(1)}(0) = \begin{bmatrix}1\\0\end{bmatrix},\ \mathbf{x}^{(2)}(0) = \begin{bmatrix}0\\1\end{bmatrix},\ \mathbf{x}^{(3)}(0) = \begin{bmatrix}3\\3\end{bmatrix}.$$

(a) Find an expression for the general solution of the system above, in terms of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$.

$c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + c_3\vec{x}^{(3)}$ where $c_1+c_2+c_3 = 1$ is always a solution. To show that these are all solutions, it is enough to check that $\vec{x}^{(1)} - \vec{x}^{(2)}$ and $\vec{x}^{(3)} - \vec{x}^{(2)}$ form a fundamental set for $\vec{x}' = A\vec{x}$. Their values at 0 are $\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}$ respectively, and these are lin. independent

(b) Find the solution of the system that satisfies the initial condition $\mathbf{x}(0) = \begin{bmatrix}1\\1\end{bmatrix}$ in terms of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$.

$$\vec{x} = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + c_3\vec{x}^{(3)}, \quad c_1+c_2+c_3 = 1.$$

We need $\vec{x}(0) = \begin{bmatrix}1\\1\end{bmatrix} = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix} + c_3\begin{bmatrix}3\\3\end{bmatrix}$

$$\Rightarrow \left.\begin{aligned} c_1 \quad + 3c_3 &= 1 \\ c_2 + 3c_3 &= 1 \\ c_1 + c_2 + c_3 &= 1 \end{aligned}\right\} \quad \begin{aligned} c_1 = c_2 &= 2/5 \\ c_3 &= 1/5 \end{aligned} \Rightarrow \boxed{\vec{x} = \frac{2}{5}\vec{x}^{(1)} + \frac{2}{5}\vec{x}^{(2)} + \frac{1}{5}\vec{x}^{(3)}}$$

Question 4 (15 pts) Solve the initial value problem $\mathbf{x}' = \underbrace{\begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}}_{A} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

where $\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda) + 5$$
$$= \lambda^2 + 2\lambda + 2$$
$$= (\lambda+1)^2 + 1$$

Eigenvalues are $\lambda_{1,2} = -1 \mp i$

Eigenvectors for $\lambda_1 = -1 + i$: Solve $(A - (-1+i)I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2-i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \vec{v} = \begin{bmatrix} (2+i)k \\ k \end{bmatrix} \quad k \neq 0$$

$\vec{z} = e^{(-1+i)t} \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$ is a complex valued solution.

$$\vec{z} = e^{-t}(\cos t + i \sin t)\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = e^{-t}\left(\begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} 2\sin t + \cos t \\ \sin t \end{bmatrix} \right)$$

$\vec{x}^{(1)} = Re(\vec{z})$ and $\vec{x}^{(2)} = Im(\vec{z})$ are two independent real solutions $\Rightarrow$ $\vec{x} = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$ is the general solution.

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2\sin t + \cos t \\ \sin t \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \left.\begin{array}{l} 2c_1 + c_2 = 1 \\ c_1 = 1 \end{array}\right\} \begin{array}{l} c_1 = 1 \\ c_2 = -1 \end{array}$$

$$\boxed{\vec{x}(t) = e^{-t} \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} - e^{-t} \begin{bmatrix} 2\sin t + \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} e^{-t}\cos t - 3e^{-t}\sin t \\ e^{-t}\cos t - e^{-t}\sin t \end{bmatrix}}$$

Question 5 (10 pts) Determine (without solving the equation) an interval in which the solution of the initial value problem

$$y' + (\tan t)y = \sin t, \quad y(\pi) = 0$$

is certain to exist.

This is a first order linear equation. By the existence – uniqueness thm. for linear equations, the solution exists on any open interval containing π such that $\tan t$ and $\sin t$ are continuous.

$\sin t$ is continuous for all t and $\tan t$ is continuous except at $\pi/2 + n\pi$, $n \in \mathbb{Z}$. Therefore, the largest interval that the thm. applies to is $(\frac{\pi}{2}, \frac{3\pi}{2})$.

(Any smaller ~~open~~ interval that contains π is also a correct answer)

Question 6 (25 pts) Consider the system of ODE's

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 5 & 0 & 0 \end{bmatrix}}_{A} \mathbf{x}.$$

(a) Determine the eigenvalues and the corresponding eigenvectors, as well as generalized eigenvectors, of the coefficient matrix.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 5 & 0 & -\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 0 & -\lambda \end{vmatrix} = (1-\lambda)^2\cdot(-\lambda)$$

Eigenvalues are $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 0$.

Eigenvectors for 1: Solve $(A-I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 5 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{v} = \begin{bmatrix} 0 \\ k \\ 0 \end{bmatrix} \quad k \neq 0 \text{ are eigenvectors.}$$

Generalized eigenvectors for 1: Solve $(A-I)\vec{v} = \begin{bmatrix} 0 \\ k \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & k \\ 5 & 0 & -1 & 0 \end{array}\right] \quad \left.\begin{array}{l} v_3 = 5v_1 \\ v_1 + 2v_3 = k \end{array}\right\} 11v_1 = k \quad \begin{array}{l} v_1 = k/11 \\ v_3 = 5k/11 \end{array} \quad \vec{v} = \begin{bmatrix} k/11 \\ \ell \\ 5k/11 \end{bmatrix} \quad \begin{array}{l} k \neq 0 \\ \ell \text{ free.} \end{array}$$

Eigenvectors for 0:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 5 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} v_1 = 0 \\ v_2 = -2v_3 \end{array} \quad \vec{v} = \begin{bmatrix} 0 \\ -2k \\ k \end{bmatrix}, \ k \neq 0$$

(b) Find the general solution of the given system.

The Jordan form of A is $J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and by part (a), $P = \begin{bmatrix} 0 & 1 & 0 \\ 11 & 0 & -2 \\ 0 & 5 & 1 \end{bmatrix}$ satisfies $AP = PJ$.

Set $\vec{x} = P\vec{y}$. Then $\vec{y}\,' = J\vec{y} \Rightarrow \left.\begin{array}{l} y_1' = y_1 + y_2 \\ y_2' = y_2 \\ y_3' = 0 \end{array}\right\} \begin{array}{l} y_1 = c_2 te^t + c_1 e^t \\ y_2 = c_2 e^t \\ y_3 = c_3 \end{array}$

$$\vec{y} = \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} 0 & 1 & 0 \\ 11 & 0 & -2 \\ 0 & 5 & 1 \end{bmatrix}\begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

(c) Determine all possible initial conditions $\mathbf{x}(0)$ such that for the solution $\mathbf{x}(t)$ of the corresponding initial value problem, $\lim_{t\to+\infty} \mathbf{x}(t)$ exists and is finite.

$$\vec{x}(t) = \begin{bmatrix} 0 & e^t & 0 \\ 11e^t & 11te^t & -2 \\ 0 & 5e^t & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1\begin{bmatrix} 0 \\ 11e^t \\ 0 \end{bmatrix} + c_2\begin{bmatrix} e^t \\ 11te^t \\ 5e^t \end{bmatrix} + c_3\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Since $e^t \to +\infty$ when $t \to +\infty$, we must have $c_2 = 0$ and $c_1 = 0$ in order to have a finite limit.

In that case, $\vec{x}(0) = c_3\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ (so $\vec{x}(0)$ should be proportional to $\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$)