MATH 219

Fall 2020

Lecture 3

Lecture notes by Özgür Kişisel

Content: Modeling with first order equations. (section 2.3)).

Suggested Problems: (Boyce, Di Prima, 9th edition)

§2.3: 4, 5, 6, 8, 12, 14, 18, 21, 23, 26, 29

1 Modeling with first order equations

The subject of differential equations is essentially as old as calculus, with Newton, Leibniz and mathematicians from the Bernoulli family among the first contributors. The growth of the subject went hand in hand with applications. It is fair to say that differential equations were among the key mathematical tools for the scientific discoveries that took place before and during the industrial revolution.

We will discuss several different problems below that can be modeled by first order ODE's. There are some common features of the process of attaching a differential equation to a given problem. Among these are:

- Determining the dependent and independent variables,
- Relating the rates of change in the problem to the variables,
- Finding the initial conditions

Once the ODE and the initial conditions are set, the next step is to solve the initial value problem.

Example 1.1 *The position of a particle moving in space can be described using three coordinate functions, each depending on a time parameter, say* $(x(t), y(t), z(t))$*. For simplicity, let us assume that only one of these three functions is changing during the motion of the particle, for instance we may imagine that the particle is moving* *along the x-axis. The first derivative of $x(t)$ is the **velocity** of the particle, and the second derivative is its **acceleration**. In other words,*

$$v(t) = x'(t), \quad a(t) = v'(t) = x''(t)$$

Newton's second law of motion indirectly describes how $x(t)$ changes. It says that the acceleration of a particle is directly proportional to the total force applied to it. The proportionality constant is called the mass of the object. The corresponding equation is the famous

$$F = ma.$$

Rewriting this law in terms of $v(t)$ or $x(t)$, we obtain the differential equations

$$v'(t) = F(t)/m, \qquad x''(t) = F(t)/m.$$

(Throughout, we assume that the mass of the object does not change with respect to time.) Of course, the actual nature of these equations highly depend on the function $F(t)$. As a first case, let us assume that $F(t)$ is constant. This happens, for instance, for a freely falling body which is close to the surface of the earth. If $F(t)$ is constant, then $a = a(t)$ will be a constant. We have

$$\begin{aligned} v'(t) &= a \\ v(t) &= v(t_0) + \int_{t_0}^{t} a d\tau \\ v(t) &= v(t_0) + a(t - t_0) \end{aligned}$$

Now we can solve for $x(t)$:

$$\begin{aligned} x'(t) &= v(t) \\ x(t) &= x(t_0) + \int_{t_0}^{t} v(\tau) d\tau \\ x(t) &= x(t_0) + \int_{t_0}^{t} v(t_0) + a(\tau - t_0) d\tau \\ x(t) &= x(t_0) + v(t_0)(t - t_0) + \frac{a(t - t_0)^2}{2} \end{aligned}$$

In this example, the differential equations were very easy to solve since their right hand sides depend only on t. Integration is sufficient for solving the problem.

Example 1.2 *Let us now assume that we have the setting in the previous example, but besides a constant force, there is an additional force on the object proportional to its velocity. Such a model may be valid, for instance, for a freely falling object subject to air resistance. An object that moves faster would collide with more air molecules per unit time, so we can imagine that the restraining force will be increased when velocity is increased. It is a strong assumption that the relation between this force and the velocity is linear - actually, in many problems of aerodynamics, a non-linear relation would be more relevant. However, we assume that the linear relation holds for the sake of simplicity. Therefore the total force on the object is a function of the form* $F_0 - kv(t)$*, where* k *is a positive constant. The differential equation for the velocity is*

$$v'(t) = \frac{F_0 - kv(t)}{m}$$

As opposed to the previous example, the right hand side contains the dependent variable now, so we cannot directly integrate. However, the equation is separable. Suppose that $v(t_0) = v_0$*.*

$$\begin{aligned}
\int_{v_0}^{v} \frac{d\nu}{F_0 - k\nu} &= \int_{t_0}^{t} \frac{d\tau}{m} \\
-\frac{1}{k} \ln\left(\frac{F_0 - kv}{F_0 - kv_0}\right) &= \frac{t - t_0}{m} \\
F_0 - kv &= (F_0 - kv_0)e^{-\frac{k}{m}(t-t_0)} \\
v &= \frac{F_0}{k} + \left(v_0 - \frac{F_0}{k}\right) e^{-\frac{k}{m}(t-t_0)}
\end{aligned}$$

Notice that for $t = t_0$ *the formula indeed gives back* v_0*, which is a quick check on the computation. The resulting function for* v *is the sum of a constant term and a decaying exponential. Therefore, for* t *large enough we expect the velocity* v *to get close to a "limiting velocity", with value* F_0/k*.*

Example 1.3 *Understanding the population dynamics of a given species is a problem of great interest in biological and environmental sciences. As a simplest possible model, suppose that the rate of growth of the population of the species is directly proportional to the existing population. This model could be valid in a case where the number of offsprings per unit time is proportional to the population and there are no other restraining factors such as a competing species or scarcity of food. Then,* *the differential equation for the population* $P(t)$ *at time* t *should be of the form:*

$$P'(t) = kP(t)$$

for a certain positive constant k. *This is a separable equation again. Assume that* $P(t_0) = P_0$.

$$\begin{aligned}\int_{P_0}^{P} \frac{d\rho}{\rho} &= \int_{t_0}^{t} k d\tau \\ \ln\left(\frac{P}{P_0}\right) &= k(t - t_0) \\ P(t) &= P_0 e^{k(t-t_0)}\end{aligned}$$

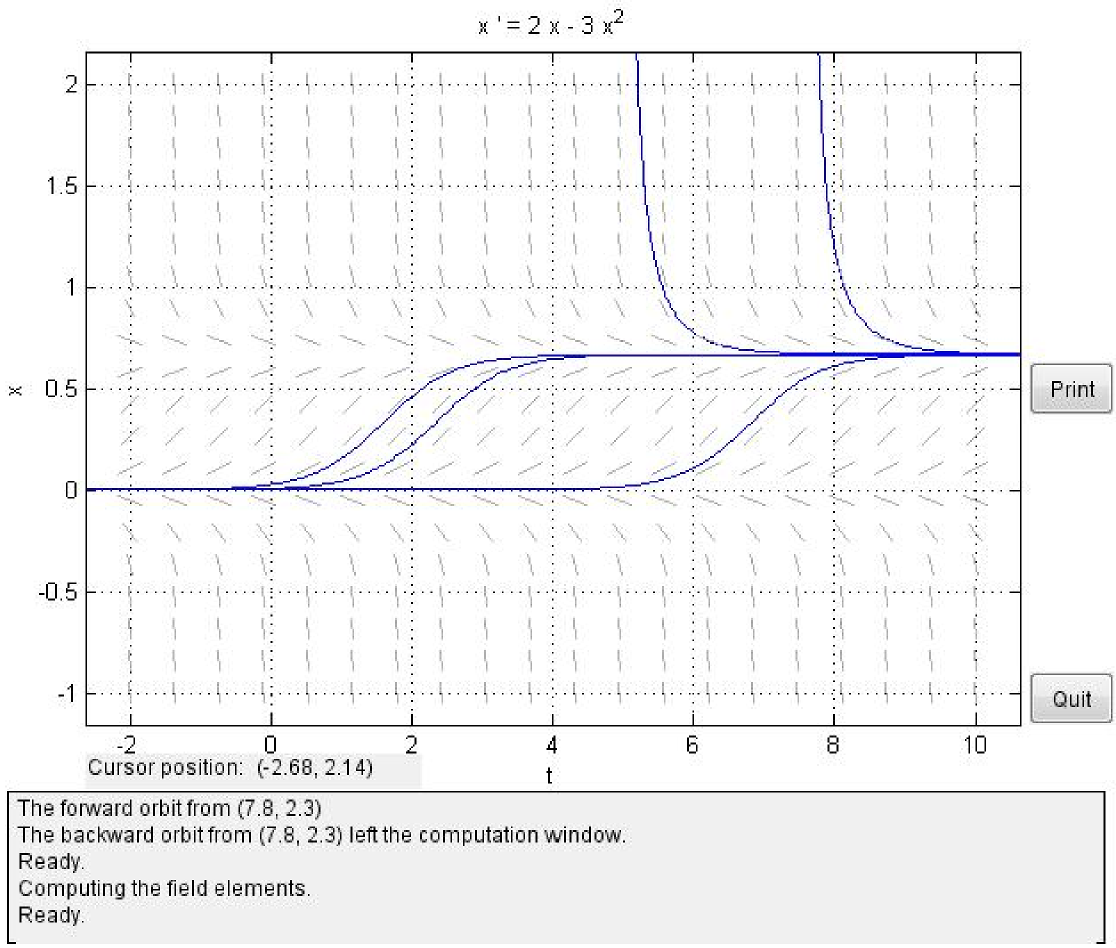
Since $k > 0$, *the resulting function* $P(t)$ *grows exponentially. Since, in a real life situation, a population cannot increase indefinitely, this model could only be valid for a limited time interval. When the population becomes large, possibly other factors (such as scarcity of food or resources) will be more dominant than the rate of reproduction and this will cause an opposite trend. Such a model can be formed by introducing a restraining factor proportional to the square of the population. Then the new differential equation is*

$$P'(t) = kP(t) - \ell P(t)^2$$

where k *and* l *are positive constants. Since this equation is separable, we can solve it by the usual methods. Instead of this, let us sketch its direction field. We took* $k = 2$ *and* $\ell = 3$ *for the figure below. Notice that the solutions are asymptotically decaying towards 0.66. In general, all solutions such that* $P_0 > 0$ *will exponentially decay towards the equilibrium solution* $P = k/\ell$.

It is not difficult to interpret the results in terms of population dynamics. If the population is initially below the critical value of k/ℓ, *then the growth term is dominant and the population begins to increase. Upon approaching* k/ℓ, *the restraint term becomes more and more dominant. The population is monotone increasing, but it cannot go above the critical value. Likewise, if the population is above the critical value, then the restraint term is dominant and the population decreases. This time, it cannot go below the critical value.*

Example 1.4 *Suppose that a savings account initially contains 1000 TL. Assume that money is compounded continuously with the same interest rate, which implies*



that the rate of change of money is proportional to the money in the account at that time. If the amount of money in the savings account is 2000 TL at the end of the second year, what will it be at the end of the fifth year?

Solution: *Let $M(t)$ denote the amount of money in the account at the end of year t. We have $M(0) = 1000$. The assumption says*

$$\frac{dM}{dt} = kM$$

for some constant k. This is a separable equation and the solutions are

$$M(t) = ce^{kt}.$$

Since $M(0) = 1000$*, we have* $c = 1000$*. In order to find the value of* k*, use the equation* $M(2) = 2000$*. Then,*

$$\begin{aligned} 2000 &= 1000e^{2k} \\ k &= \frac{\ln 2}{2} \end{aligned}$$

Therefore the amount of money at the end of year t *is given by the formula*

$$M(t) = 1000e^{(\ln 2)t/2} = 1000(2^{t/2})$$

Finally, $M(5) = 1000(2^{5/2}) = 5656.85$ *TL.*

Example 1.5 *At time* $t = 0$ *a tank contains 50 kg of salt dissolved in 1000* ℓ *of water. Assume that water containing 100 g of salt per liter is entering the tank at a rate of* $30\ell/min$ *and that the well-stirred mixture is leaving the tank at the same rate. Find the limiting amount of salt that is present in the tank after a very long time.*

Solution: *Let* $Q(t)$ *denote the amount of salt (in kilograms) in the tank at time* t*. Then the rate of change of* $Q(t)$ *can be computed as the difference of the rate of incoming salt and the rate of outgoing salt. The incoming rate is constant, namely* $0.1kg/\ell \times 30\ell/min = 3kg/min$*. However, the outgoing rate is not constant. It changes with the concentration. The concentration of salt at time* t *is* $Q(t)/1000$*. Therefore the outgoing rate must be* $30Q/1000$*.*

$$\begin{aligned} \frac{dQ}{dt} &= 3 - \frac{3Q}{100} \\ \int \frac{dQ}{300 - 3Q} &= \int \frac{dt}{100} \\ -\frac{1}{3}\ln|300 - 3Q| &= \frac{t}{100} + c \\ 300 - 3Q &= ce^{-\frac{3t}{100}} \\ Q(t) &= 100 + ce^{-\frac{3t}{100}} \end{aligned}$$

We can find the constant c *by using the value of* $Q(0)$ *but the answer to the question is* $\lim_{t\to\infty} Q(t) = 100kg$*, which is independent of the value of* c*.*

Example 1.6 *A tank initially contains 60 ℓ of pure water. A solution containing 1 g of salt per liter enters the tank at 2ℓ/min, and the perfectly mixed solution leaves the tank at 3ℓ/min, therefore the tank is empty after 1 hour.*

(a) Express the volume of solution in the tank in terms of t and write a differential equation for the amount of salt in the tank at time t.

(b) Find the amount of salt in the tank after t minutes for $t \leq 60$.

(c) What is the maximum amount of salt ever in the tank?

Solution: *(a)* $V(t) = 60 - t$. *Let* $Q(t)$ *denote the amount of salt in the tank at time* t.

$$\frac{dQ}{dt} = 2 - \frac{3Q}{60 - t}.$$

(b) Let us rewrite the equation as $\frac{dQ}{dt} + \frac{3Q}{60 - t} = 2$. *This is a first order linear equation with integrating factor* $\mu(t) = exp(\int 3(60-t)^{-1}dt) = (60-t)^{-3}$. *Multiplying the ODE by* $\mu(t)$, *we get*

$$\begin{aligned}\frac{d((60-t)^{-3}Q)}{dt} &= 2(60-t)^{-3}\\ (60-t)^{-3}Q &= (60-t)^{-2} + c\\ Q &= 60 - t + c(60-t)^3\end{aligned}$$

Since $Q(0) = 0$ *we have* $c = -60^{-2}$. *So*

$$Q = 60 - t - \frac{(60-t)^3}{60^2}$$

(c) At $t = 0$ *and* $t = 60$ *we have* $Q = 0$ *therefore* Q *attains its maximum at an interior point* t^* *of the interval* $[0, 60]$. *At that point* $Q'(t^*) = 0$, *so*

$$\begin{aligned}-1 + \frac{3(60-t^*)^2}{60^2} &= 0\\ 60 - t^* = \sqrt{1200} &= 34.64\\ t^* &= 25.36\\ Q(t^*) &= 23.09g\end{aligned}$$