MATH 219

Fall 2020

Lecture 22

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Content: Solution of Initial Value Problems. Step Functions

Suggested Problems (Boyce, Di Prima, 9th edition):

§6.2: 1, 4, 7, 9, 12, 16, 17

§6.3: 1, 3, 8, 10, 14, 20, 23, 38

Let us now return to differential equations, after our general discussion about Laplace transforms. Suppose that we have a linear ordinary differential equation in $y(t)$ together with initial conditions. By taking the Laplace transforms of both sides of the equation, we end up with an equation for the Laplace transform $Y(s)$. In favourable cases, this equation is easier to solve than the original equation, or of equal difficulty. After finding $Y(s)$, one may find $y(t)$ provided that one can compute the inverse of the Laplace transform operation in this particular case. This procedure works especially well if the original linear ODE has constant coefficients. Let us now work out the details of this procedure. We first have to understand what the effect of Laplace transform is on the derivative of a function.

1 Laplace Transform of a Derivative

Theorem 1.1 *Suppose that $f(t)$ is continuous and $f'(t)$ is piecewise continuous on $[0,\infty)$. Furthermore suppose that there exists constants K, a such that $|f(t)| \leq Ke^{at}$ for all t. Then $\mathcal{L}\{f'(t)\}$ exists for all $s > a$ and*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Proof Recall that $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)dt = \lim_{M\to\infty}\int_0^M e^{-st}f'(t)dt$. Since $f'(t)$ is piecewise continuous on $[0, M]$, so is $e^{-st}f'(t)$. Say that the points of discontinuity are $t_1, \ldots, t_n$. Then we can break up the last integral into pieces:

$$\int_0^M e^{-st}f'(t)dt = \int_0^{t_1} e^{-st}f'(t)dt + \int_{t_1}^{t_2} e^{-st}f'(t)dt + \ldots + \int_{t_{n-1}}^{t_n} e^{-st}f'(t)dt + \int_{t_n}^M e^{-st}f'(t)dt.$$

Now, each of the integrands on the right hand side is continuous on the relevant interval. Therefore we can apply integration by parts to each of the integrals on the right hand side. Set $u = e^{-st}$ and $dv = f'(t)dt$. Then $du = -se^{-st}$ and $v = f(t)$. Use $\int_a^b udv = uv|_a^b - \int_a^b vdu$ for each integrand on the right hand side and get

$$\begin{aligned}\int_0^M e^{-st} f'(t)dt = \quad & e^{-st}f(t)|_0^{t_1} - \int_0^{t_1} -se^{-st}f(t)dt + e^{-st}f(t)|_{t_1}^{t_2} - \int_{t_1}^{t_2} -se^{-st}f(t)dt \\ & + \ldots + e^{-st}f(t)|_{t_n}^{M} - \int_{t_n}^{M} -se^{-st}f(t)dt \\ & = s\int_0^M e^{-st}f(t)dt + e^{-sM}f(t) - f(0).\end{aligned}$$

It remains to send M to ∞. The inequality $|f(t)| \leq Ke^{at}$ implies that $e^{-sM}f(t)$ tends to 0 when $M \to \infty$. Therefore we obtain

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \qquad \square$$

Let us assume for a moment that derivatives of $f(t)$ of all orders satisfy similar conditions to those in the theorem without spelling them out loud. In this case, we can apply the theorem to derivatives of $f(t)$ recursively and get,

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \\ \mathcal{L}\{f'''(t)\} &= s^3\mathcal{L}\{f(t)\} - s^2f(0) - sf'(0) - f''(0).\end{aligned}$$

and so on. More precisely, we have:

Theorem 1.2 *Suppose that $f(t), f'(t) \ldots f^{(n-1)}(t)$ are continuous and that $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$. Suppose that there exists constants K, a such that $|f^{(i)}(t)| \leq Ke^{at}$ for all $0 \leq i \leq n-1$. Then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and*

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

The proof of this theorem follows directly from the previous one by applying induction. Notice that in the folrmula, the first term is s^n times the Laplace transform of $f(t)$. The other terms are related to the initial conditions.

2 Obtaining and Solving the Equation for $Y(s)$

Suppose now that we have a linear ODE for $y(t)$. Apply $\mathcal{L}$ to both sides of the equation and use the linearity of $\mathcal{L}$. This gives us an equation for $Y(s) = \mathcal{L}\{y(t)\}$. The aim is then to solve the equation for $Y(s)$. Let us demonstrate this procedure through a few examples.

Example 2.1 *Suppose that $y'' - 5y' + 4y = \cos 3t$, furthermore suppose that $y(0) = 2$ and $y'(0) = 0$. Find $Y(s)$, the Laplace transform of $y(t)$.*

Solution: *Apply $\mathcal{L}$ to both sides of the equation:*

$$\begin{aligned}
\mathcal{L}\{y'' - 5y' + 4y\} &= \mathcal{L}\{\cos 3t\} \\
\mathcal{L}\{y''\} - 5\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} &= \frac{s}{s^2+9} \\
s^2Y(s) - sy(0) - y'(0) - 5(sY(s) - y(0)) + 4Y(s) &= \frac{s}{s^2+9} \\
(s^2 - 5s + 4)Y(s) &= \frac{s}{s^2+9} + sy(0) + y'(0) - 5y(0) \\
Y(s) &= \frac{s}{(s^2+9)(s^2-5s+4)} + \frac{2s-10}{s^2-5s+4}.
\end{aligned}$$

Soon, we will discuss how one can recover the solution $y(t)$ from $Y(s)$.

Example 2.2 *Suppose that $y^{(4)} + y = e^{-2t}$, furthermore suppose that $y(0) = 1, y'(0) = 0, y''(0) = 2, y'''(0) = 0$. Find the Laplace transform $Y(s)$ of $y(t)$.*

Solution: *Apply $\mathcal{L}$ to both sides of the equation:*

$$\begin{aligned}
\mathcal{L}\{y^{(4)} + y\} &= \mathcal{L}\{e^{-2t}\} \\
\mathcal{L}\{y^{(4)}\} + \mathcal{L}\{y\} &= \frac{1}{s+2} \\
s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) + Y(s) &= \frac{1}{s+2} \\
(s^4+1)Y(s) &= \frac{1}{s+2} + s^3 + 2s \\
Y(s) &= \frac{1}{(s+2)(s^4+1)} + \frac{s^3+2s}{s^4+1}.
\end{aligned}$$

3 Finding $y(t)$, When $Y(s)$ is Given

Our next question is whether or not we can recover $y(t)$ from $Y(s)$. We will state the following theorem without proof:

Theorem 3.1 *Suppose that $f(t)$ and $g(t)$ are two piecewise continuous functions on $[0, \infty)$ and there exist constants K, a such that $|f(t)| \leq Ke^{at}$ and $|g(t)| \leq Ke^{at}$. If $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$ for all $s > a$, then $f(t) = g(t)$ at all points for which f and g are both continuous.*

In particular, the only possible places that $f \neq g$ can be the points of discontinuity of f or g. In every finite interval, we have only finitely many such points. So f and g are equal "almost everywhere". The Laplace transform is after all an integral transformation, so it shouldn't be expected to see the differences between two functions at finitely many points. So in a sense, this result is the best that one could expect.

If $\mathcal{L}\{y(t)\} = Y(s)$, then we will write $y(t) = \mathcal{L}^{-1}\{Y(s)\}$, ignoring the problem about discontinuities mentioned in the previous paragraph. The transformation $\mathcal{L}^{-1}$ is called the **inverse Laplace transform**. It is again linear. Just like the integral formula for the Laplace transform, there is a similar integral formula for the inverse Laplace transform. However, it requires the use of a line integral in the complex plane and its evaluation often uses techniques from complex analysis involving residue calculations. Since we are not assuming that the reader has this background, we will do something else to compute $\mathcal{L}^{-1}\{Y(s)\}$ when $Y(s)$ is "simple enough".

Suppose that $Y(s)$ is a rational function, namely $Y(s) = P(s)/Q(s)$ where P and Q are both polynomials. We can break up this rational function into its partial fractions. We can then use the linearity of $\mathcal{L}^{-1}$: Find $\mathcal{L}^{-1}$ of each simple piece and add them up.

Example 3.1 *Find the inverse Laplace transform of* $\dfrac{1}{s^2 - 5s + 4}$.

Solution:

$$\frac{1}{s^2 - 5s + 4} = \frac{1}{(s-4)(s-1)} = \frac{A}{s-4} + \frac{B}{s-1}.$$

Upon equating denominators, we get $(A+B)s+(-4A-B)=1$. Since this equality should hold as an equality of polynomials,

$$\begin{aligned} A+B &= 0 \\ -4A-B &= 1. \end{aligned}$$

From here we find that $A=-1/3$ and $B=1/3$. Therefore,

$$\begin{aligned} \frac{1}{s^2-5s+4} &= \frac{-1/3}{s-4}+\frac{1/3}{s-1} \\ \mathcal{L}^{-1}\left\{\frac{1}{s^2-5s+4}\right\} &= -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}+\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\ &= -\frac{1}{3}e^{4t}+\frac{1}{3}e^{t}. \end{aligned}$$

Example 3.2 *Solve the initial value problem $y'''-y=1$, with the initial conditions $y(0)=y'(0)=y''(0)=0$.*

Solution: *Apply $\mathcal{L}$ to both sides.*

$$\begin{aligned} \mathcal{L}\{y'''-y\} &= \mathcal{L}\{1\} \\ \mathcal{L}\{y'''\}-\mathcal{L}\{y\} &= \frac{1}{s} \\ (s^3-1)Y(s) &= \frac{1}{s} \\ Y(s) &= \frac{1}{s(s^3-1)}. \end{aligned}$$

Now let us break up the right hand side into its partial fractions:

$$\frac{1}{s(s^3-1)}=\frac{A}{s}+\frac{B}{s-1}+\frac{Cs+D}{s^2+s+1}$$

Upon equating denominators, we get $1=A(s^3-1)+B(s^2+s+1)s+(Cs+D)(s-1)s$. From here, we get

$$\begin{aligned} A+B+C &= 0 \\ B-C+D &= 0 \\ B-D &= 0 \\ -A &= 1. \end{aligned}$$

The solution is $A = -1, B = 1/3, C = 2/3, D = 1/3$. *Therefore,*

$$\begin{aligned} Y(s) &= \frac{1}{s(s^3-1)} = \frac{-1}{s} + \frac{1/3}{s-1} + \frac{2s/3+1/3}{s^2+s+1} \\ &= \frac{-1}{s} + \frac{1/3}{s-1} + \frac{2(s+1/2)/3}{(s+1/2)^2+3/4}. \end{aligned}$$

Finally, applying $\mathcal{L}^{-1}$ *to both sides, we get*

$$y(t) = -1 + \frac{e^t}{3} + \frac{2}{3}e^{-t/2}\cos(\sqrt{3}t/2).$$

4 Step functions

Laplace transform is especially handy for solving ODE's with a discontinuous right hand side. We would now like to express discontinuous functions in terms of certain basic ones and extend our Laplace transform computations to them.

Definition 4.1 *Suppose that* $c \geq 0$. *The* ***unit step function at time*** $t = c$ *is*

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases}$$

Notice that $u_c(t)$ is piecewise continuous and its only discontinuity is at time $t = c$.

Suppose now that $g(t)$ is a piecewise continuous function on $\mathbb{R}_{\geq 0}$, with discontinuities at the points $0 < c_1 < c_2 < c_3 < \ldots$ More precisely, suppose that we have a piecewise definition of $g(t)$ of the type:

$$g(t) = \begin{cases} h_1(t), & t < c_1, \\ h_2(t), & c_1 \leq t < c_2, \\ h_3(t), & c_2 \leq t < c_3, \\ \ldots \end{cases}$$

We claim that

$$g(t) = h_1(t) + u_{c_1}(t)(h_2(t) - h_1(t)) + u_{c_2}(t)(h_3(t) - h_2(t)) + u_{c_3}(t)(h_4(t) - h_3(t)) + \ldots$$

This can be verified easily: Just check that the left and right hand sides agree for all values of t. First, suppose that $t < c_1$. Then only the first summand in the right hand side is nonzero and it is $h_1(t)$. Next, suppose that $c_1 \leq t < c_2$. Then $u_{c_1}(t) = 1$ but all other $u_{c_j}(t) = 0$ for $j \geq 2$. Thus we get $h_1(t) + (h_2(t) - h_1(t)) = h_2(t)$ on the right hand side. Progressing in this way, if $c_i \leq t < c_{i+1}$, then $u_{c_1}(t) = \ldots = u_{c_i}(t) = 1$ and $u_{c_{i+1}}(t) = u_{c_{i+2}}(t) = \ldots = 0$. Therefore, the right hand side becomes

$$h_1(t) + (h_2(t) - h_1(t)) + (h_3(t) - h_2(t)) + \ldots + (h_{i+1}(t) - h_i(t)) = h_{i+1}(t).$$

This agrees with the given expression for $g(t)$.

Example 4.1 *Suppose that $g(t)$ is the function given in piecewise form by*

$$g(t) = \begin{cases} 0, & t < 2, \\ t-2, & 2 \leq t < 3, \\ 4-t, & 3 \leq t < 4, \\ 0, & t \geq 4. \end{cases}$$

By the argument above, we have

$$\begin{aligned} g(t) &= 0 + u_2(t)(t-2-0) + u_3(t)(4-t-(t-2)) + u_4(t)(0-(4-t)) \\ &= (t-2)u_2(t) + (6-2t)u_3(t) + (t-4)u_4(t). \end{aligned}$$

Example 4.2 *Suppose that $g(t)$ is the square-wave function whose value is 1 if $2n \leq t < 2n+1$ and 0 if $2n-1 \leq t < 2n$ for every nonnegative integer n. Notice that this time $g(t)$ has infinitely many discontinuities, at positive integer values of t.*

$$\begin{aligned} g(t) &= 1 + (0-1)u_1(t) + (1-0)u_2(t) + (0-1)u_3(t) + \ldots \\ &= 1 - u_1(t) + u_2(t) - u_3(t) + u_4(t) - \ldots \end{aligned}$$

Now let us see how we can compute Laplace transforms of expressions containing step functions. As a preliminary step we will calculate the Laplace transform of a step function itself.

$$
\begin{aligned}
\mathcal{L}\{u_c(t)\} &= \int_0^\infty u_c(t)e^{-st}dt \\
&= \int_c^\infty e^{-st}dt \\
&= \int_0^\infty e^{-s(\tau+c)}d\tau \\
&= e^{-sc}\int_0^\infty e^{-s\tau}d\tau \\
&= e^{-sc}\mathcal{L}\{1\} \\
&= \frac{e^{-sc}}{s}.
\end{aligned}
$$

where the transform converges for $Re(s) > 0$. The substitution $\tau = t - c$ was used in the second step.

More or less the same computation allows us to prove the following theorem which vastly generalizes the result above:

Theorem 4.1 *Suppose that $c > 0$ and the Laplace transform of $f(t)$ converges for $s > a$. Then*

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-sc}\mathcal{L}\{f(t)\}$$

for $s > a$.

Proof

$$
\begin{aligned}
\mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty u_c(t)f(t-c)e^{-st}dt \\
&= \int_c^\infty f(t-c)e^{-st}dt \\
&= \int_0^\infty f(\tau)e^{-s(\tau+c)}d\tau \\
&= e^{-sc}\int_0^\infty f(\tau)e^{-s\tau}d\tau \\
&= e^{-sc}\mathcal{L}\{f(t)\}.
\end{aligned}
$$

$\square$

Example 4.3 *Suppose that* $g(t)$ *is as in example 4.1, namely* $g(t) = (t-2)u_2(t) + (6-2t)u_3(t) + (t-4)u_4(t)$*. Using the theorem,*

$$\mathcal{L}\{g(t)\} = \frac{e^{-2s}}{s^2} - 2\frac{e^{-3s}}{s^2} + \frac{e^{-4s}}{s^2}.$$

Example 4.4 *Next, suppose that* $g(t)$ *is as in example 4.2, therefore* $g(t) = 1 - u_1(t) + u_2(t) - u_3(t) + u_4(t) = \ldots$ *Again, using the theorem,*

$$\begin{aligned}\mathcal{L}\{g(t)\} &= \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \ldots \\ &= \frac{1}{s(1+e^{-s})}.\end{aligned}$$

The last equality uses the formula for geometric series and is valid for $Re(s) > 0$*.*