MATH 219

Fall 2020

Lecture 20

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Content: Series Solutions Near A Regular Singular Point

Suggested Problems: (Boyce, Di Prima, 9th edition)

§5.5: 4,10,11,12,13

Suppose that x_0 is a regular singular point for the equation

$$p(x)y'' + q(x)y' + r(x)y = 0 \qquad (*)$$

Also, suppose that

$$\lim_{x \to x_0} \frac{(x - x_0)q(x)}{p(x)} = \alpha, \qquad \lim_{x \to x_0} \frac{(x - x_0)^2 r(x)}{p(x)} = \beta.$$

Then we regard the Cauchy-Euler equation

$$(x - x_0)^2 y'' + \alpha(x - x_0)y' + \beta y = 0 \qquad (**)$$

as being "close" to the equation $(*)$ in the sense that the first Taylor series terms of the coefficients of the two ODE's agree. We saw before how one can solve Cauchy-Euler equations. A reasonable guess is that if we perturb a solution of the Cauchy-Euler equation $(**)$ by multiplying it with an appropriate power series, then we can obtain a solution for $(*)$. This strategy turns out to be reasonably succesful, as detailed below:

<u>Strategy for solving an ODE near a regular singular point</u>

- Check that x_0 is a regular singular point and find the limits α, β above.
- Find the roots r_1, r_2 of the indicial equation $r^2 + (\alpha - 1)r + \beta = 0$. We will assume that the roots are real for the sake of simplicity, but the complex case is also manageable.

- Say $r_1 \geq r_2$. If $r_1 - r_2$ is not an integer, then one can obtain two linearly independent power series solutions for (*)

$$y_1 = |x - x_0|^{r_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad y_2 = |x - x_0|^{r_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

- If $r_1 - r_2$ is an integer, then a solution y_1 as above still exists, but y_2 above need not exist. Instead, usually one will need to have a logarithmic term in the second solution (we will not cover the details for such a second solution here).

- In the solutions above, the coefficients a_0 and b_0 will be free, so they can be taken to be 1 without loss of generality.

Example 0.1 *Solve the equation $2x^2y'' + 3xy' + (2x^2 - 1)y = 0$, centered at $x_0 = 0$.*

Solution: *The function $3x/2x^2$ is not analytic at 0, therefore $x_0 = 0$ is not an ordinary point. The functions $x \cdot 3x/2x^2$ and $x^2 \cdot (2x^2 - 1)/2x^2$ are both analytic near 0, so the singularity is regular. The limits of the two functions are $\alpha = 3/2$ and $\beta = -1/2$ respectively.*

The indicial equation is

$$r^2 + \frac{1}{2}r - \frac{1}{2} = 0$$

The two roots of this equation are $r_1 = \frac{1}{2}$ and $r_2 = -1$. Their difference is not an integer, so we should have two linearly independent solutions of the form

$$y_1 = |x|^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n x^n, \qquad y_2 = |x|^{-1} \sum_{n=0}^{\infty} b_n x^n$$

Let us assume from now on that $x > 0$, so that we can remove the absolute values. The case $x < 0$ is similar.

$$\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\
y_1' &= \sum_{n=0}^{\infty} (n + \frac{1}{2}) a_n x^{n-\frac{1}{2}} \\
y_1'' &= \sum_{n=0}^{\infty} (n + \frac{1}{2})(n - \frac{1}{2}) a_n x^{n-\frac{3}{2}}
\end{aligned}$$

Notice that the initial terms in the sums are non-constant, hence they should be still kept after taking derivatives. Putting these terms in the ODE, we get

$$2x^2\sum_{n=0}^{\infty}(n+\frac{1}{2})(n-\frac{1}{2})a_nx^{n-\frac{3}{2}}+3x\sum_{n=0}^{\infty}(n+\frac{1}{2})a_nx^{n-\frac{1}{2}}+(2x^2-1)\sum_{n=0}^{\infty}a_nx^{n+\frac{1}{2}}=0$$

$$\sum_{n=0}^{\infty}2(n+\frac{1}{2})(n-\frac{1}{2})a_nx^{n+\frac{1}{2}}+\sum_{n=0}^{\infty}3(n+\frac{1}{2})a_nx^{n+\frac{1}{2}}+\sum_{n=0}^{\infty}2a_nx^{n+\frac{5}{2}}-\sum_{n=0}^{\infty}a_nx^{n+\frac{1}{2}}=0$$

$$\sum_{n=0}^{\infty}2(n+\frac{1}{2})(n-\frac{1}{2})a_nx^{n+\frac{1}{2}}+\sum_{n=0}^{\infty}3(n+\frac{1}{2})a_nx^{n+\frac{1}{2}}+\sum_{n=2}^{\infty}2a_{n-2}x^{n+\frac{1}{2}}-\sum_{n=0}^{\infty}a_nx^{n+\frac{1}{2}}=0$$

$$(2\cdot\frac{1}{2}\cdot(-\frac{1}{2})\cdot a_0+3\cdot\frac{1}{2}\cdot a_0-a_0)x^{\frac{1}{2}}+(2\cdot\frac{3}{2}\cdot\frac{1}{2}a_1+3\cdot\frac{3}{2}a_1-a_1)x^{\frac{3}{2}}+$$

$$\sum_{n=2}^{\infty}[2(n+\frac{1}{2})(n-\frac{1}{2})a_n+3(n+\frac{1}{2})a_n+2a_{n-2}-a_n]x^{n+\frac{1}{2}}=0$$

We now equate the coefficient of each power of x in the above expression to 0. First of all, the coefficient of $x^{\frac{1}{2}}$ is 0, therefore a_0 is free. Next, the coefficient of $x^{\frac{3}{2}}$ shows $a_1=0$. The coefficient in the final summand gives, for $n\geq 2$,

$$(2n^2+3n)a_n+2a_{n-2}=0.$$

Therefore, we obtain the recursion relation

$$a_n=-\frac{2}{2n^2+3n}a_{n-2}.$$

We immediately get $a_1=a_3=a_5=\ldots=0$. The first few of the even indexed terms are

$$a_2=-\frac{a_0}{7},\qquad a_4=-\frac{a_2}{22}=\frac{a_0}{154},\qquad a_6=-\frac{1}{45}a_4=-\frac{a_0}{6930}$$

Taking $a_0=1$, we get

$$y_1=x^{\frac{1}{2}}\left(1-\frac{x^2}{7}+\frac{x^4}{154}-\frac{x^6}{6930}+\ldots\right)$$

We will now carry out the same steps for the second solution y_2:

$$\begin{aligned}
y_2 &= \sum_{n=0}^{\infty} b_n x^{n-1} \\
y_2' &= \sum_{n=0}^{\infty} (n-1) b_n x^{n-2} \\
y_2'' &= \sum_{n=0}^{\infty} (n-1)(n-2) b_n x^{n-3}
\end{aligned}$$

Put these terms in the ODE:

$$\begin{gathered}
2x^2 \sum_{n=0}^{\infty} (n-1)(n-2) b_n x^{n-3} + 3x \sum_{n=0}^{\infty} (n-1) b_n x^{n-2} + (2x^2-1) \sum_{n=0}^{\infty} b_n x^{n-1} = 0 \\
\sum_{n=0}^{\infty} 2(n-1)(n-2) b_n x^{n-1} + \sum_{n=0}^{\infty} 3(n-1) b_n x^{n-1} + \sum_{n=0}^{\infty} 2 b_n x^{n+1} - \sum_{n=0}^{\infty} b_n x^{n-1} = 0 \\
\sum_{n=0}^{\infty} 2(n-1)(n-2) b_n x^{n-1} + \sum_{n=0}^{\infty} 3(n-1) b_n x^{n-1} + \sum_{n=2}^{\infty} 2 b_{n-2} x^{n-1} - \sum_{n=0}^{\infty} b_n x^{n-1} = 0 \\
(2 \cdot (-1) \cdot (-2) b_0 + 3 \cdot (-1) b_0 - b_0) x^{-1} + (2 \cdot 0 \cdot (-1) b_1 + 3 \cdot 0 \cdot b_1 - b_1) x^0 + \\
\sum_{n=2}^{\infty} [2(n-1)(n-2) b_n + 3(n-1) b_n + 2 b_{n-2} - b_n] x^{n-1} = 0
\end{gathered}$$

Again, equate all coefficients of powers of x in the above expression to 0. We see that b_0 is free and $b_1 = 0$. The coefficient in the last summand gives, for $n \geq 2$,

$$(2n^2 - 3n) b_n + 2 b_{n-2} = 0.$$

Therefore, the recursion relation is

$$b_n = -\frac{2}{2n^2 - 3n} b_{n-2}.$$

We get $b_1 = b_3 = b_5 = \ldots = 0$. The first few even indexed terms are:

$$b_2 = -b_0, \qquad b_4 = -\frac{b_2}{10} = \frac{b_0}{10}, \qquad b_6 = -\frac{b_4}{27} = -\frac{b_0}{270}$$

Taking $b_0 = 1$*, we get*

$$y_2 = x^{-1}\left(1 - x^2 + \frac{x^4}{10} - \frac{x^6}{270} + \dots\right)$$

Finally, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

where c_1, c_2 *are arbitrary constants and* y_1, y_2 *are as above.*