MATH 219
Fall 2020
Lecture 2
Lecture notes by Özgür Kişisel

Content: Linear equations; method of integrating factors. (section 2.1).

Suggested Problems: (Boyce, DiPrima, 9th edition)

§2.1: 2, 5, 10, 14, 21, 28, 30, 32, 33, 36

1   Linear Equations, Integrating Factors

In this section we will study another special class of first order equations called "linear equations". We will define these equations by writing down what they look like. This way of defining them does not clarify where the name "linear" comes from. Things will be more lucid when we carefully study the notion of linearity in later sections. For now, we will focus on solving these equations.

Definition 1.1 *A first order ODE is called **linear** if it can be written in the form*

$$\frac{dy}{dt} + p(t)y = q(t)$$

for some functions $p(t)$ and $q(t)$.

Notice that in a linear equation y must appear only in a very special way. For instance, the equation

$$\frac{dy}{dt} = ty^2$$

is not linear since we have the y^2 term. On the other hand, the equation

$$\frac{dy}{dt} = ty - e^{-4t}$$

is linear with $p(t) = -t$ and $q(t) = -e^{-4t}$.

Since the right hand side of a linear equation is a function of t only, we can integrate it with respect to t. If we could also integrate the left hand side with respect to t then we would have solved the question. Usually, the left hand side won't be the derivative of a single term. But maybe after multiplying both sides with a function of t, the new left hand side may become the derivative of a single term. We explore this possibility now. Say we multiply both sides by a function $\mu(t)$. Since we don't know what function $\mu(t)$ would actually work for this purpose, let us keep it as an unknown for now and determine it later.

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)q(t).$$

Can the left hand side be equal to the derivative of $\mu(t)y$? This would require

$$(\mu(t)y)' = \mu(t)y' + \mu(t)p(t)y$$
$$\mu(t)'y + \mu(t)y' = \mu(t)y' + \mu(t)p(t)y$$
$$\mu(t)'y = \mu(t)p(t)y$$
$$\mu(t)' = \mu(t)p(t)$$

But the last equation that we obtained above is a separable differential equation for the function $\mu(t)$, therefore we should be able to solve it. Indeed,

$$\int \frac{d\mu}{\mu} = \int p(t)dt$$
$$\ln|\mu| = \int p(t)dt$$
$$\mu(t) = e^{\int p(t)dt}$$

A function $\mu(t)$ which makes the left hand side of the equation equal to the derivative of a single term after multiplication is called an **integrating factor** for the ODE. Therefore, the formula above gives us an integrating factor for any linear equation. We suppressed the $+c$ in the integral above: If we kept it, we would have obtained infinitely many integrating factors all of which are scalar multiples of one another. This is not of great importance here since if $\mu(t)$ is an integrating factor then clearly so is any of its constant multiples.

With this choice of $\mu(t)$, we can then proceed to solve the ODE and even obtain an

explicit formula for the solution, involving integrals:

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)q(t)$$

$$\frac{d(\mu(t)y)}{dt} = \mu(t)q(t)$$

$$\mu(t)y = \int \mu(t)q(t)dt$$

$$y = \frac{\int \mu(t)q(t)dt}{\mu(t)}$$

This formula is great, we can even understand how y depends on the functions $p(t)$ and $q(t)$ and we can estimate the effects of changing them. Since $\mu(t)$ is never zero, the final division operation is not a problem. If $p(t)$ and $q(t)$ are integrable functions then all the integrals in this formula exist - at least at a theoretical level. It is a whole different question whether or not one can do the integrations explicitly by pencil and paper, or by a computer.

Example 1.1 *Find all solutions of the equation*

$$y' - y - te^t = 0.$$

Solution: *Let us first write the equation in the form*

$$y' - y = te^t.$$

Therefore the equation is linear with the choices of $p(t) = -1$ and $q(t) = te^t$. As in the discussion above, set

$$\mu(t) = e^{\int p(t)dt} = e^{\int -1dt} = e^{-t}.$$

If we multiply the ODE by $\mu(t)$ and follow the steps above, we get

$$e^{-t}y' - e^{-t}y = t$$

$$(e^{-t}y)' = t$$

$$e^{-t}y = \int tdt = \frac{t^2}{2} + c$$

$$y = \frac{t^2e^t}{2} + ce^t$$

Alternatively, one can directly use the formula, but it is not as much fun.

Example 1.2 *Find a solution of the differential equation*

$$ty' + 3y = \frac{2}{t}$$

that satisfies the initial condition $y(1) = 2$.

Solution: *First, let us divide by* t *and rewrite the equation as*

$$y' + \frac{3}{t}y = \frac{2}{t^2},$$

otherwise the equation would not fit into the scheme above. Therefore the equation is linear with $p(t) = 3/t$ *and* $q(t) = 2/t^2$. *Set*

$$\mu(t) = exp\left(\int p(t)dt\right) = exp\left(\int \frac{3}{t}dt\right) = exp(3\ln t) = t^3.$$

(The diligent reader may worry about the absolute value in the natural logarithm after integration. This does not cause a problem essentially because both $\pm t^3$ *are integrating factors. The equation is problematic at* $t = 0$ *anyway, so the intervals* $(0, \infty)$ *and* $(-\infty, 0)$ *have to be treated separately.) Next, we multiply the ODE with* $\mu(t)$ *and we have*

$$t^3 y' + 3t^2 y = 2t$$
$$(t^3 y)' = 2t$$
$$t^3 y = t^2 + c.$$

In order to determine c, *we can use the initial condition now:*

$$1^3 y(1) = 1^2 + c,$$

therefore $c = 1$. *Finally, the solution is*

$$y = \frac{1}{t} + \frac{1}{t^3}.$$

Example 1.3 *Find all solutions of the equation*

$$\frac{dy}{dx} = \frac{y}{x + y^2}$$

Solution: *The equation is not linear as given. However, let us rewrite it as if y is the independent variable and x the dependent variable instead:*

$$\frac{dx}{dy} = \frac{x + y^2}{y}$$

$$\frac{dx}{dy} - \frac{x}{y} = y.$$

In this form, the equation is linear with $\mu(y) = exp(\int -1/y\,dy) = y^{-1}$. We then follow the usual steps:

$$y^{-1}\frac{dx}{dy} - \frac{x}{y^2} = 1$$

$$(y^{-1}x)' = 1$$

$$y^{-1}x = y + c$$

$$x = y^2 + cy.$$

If one wishes to obtain an expression for y in terms of x rather than one for x in terms of y, then it would be necessary to solve a quadratic equation. We leave this part as an exercise. One also has to be careful with the division in the beginning and recover the "lost" solution $y = 0$.

2 Exponential growth and decay

An extremely important special case of linear equations is when the coefficient in front of the dependent variable is a constant. So, let us say that our equation is of the form

$$\frac{dy}{dt} + ay = q(t).$$

In this case, $\mu(t) = e^{at}$, therefore

$$(e^{at}y)' = e^{at}q(t)$$

$$e^{at}y = \int e^{at}q(t)dt$$

$$y = e^{-at}\int e^{at}q(t)dt$$

If $q(t)$ is simple enough, then we can carry out the integration explicitly and obtain all solutions. We can sometimes deduce results about families of equations, as the next example shows:

Example 2.1 *Let $a > 0$. Show that all solutions of the differential equation*

$$y' + ay = b$$

have the same limit as $t \to \infty$, given that $a > 0$ and b are constants.

Solution: *Let us trace the steps of the solution once again. The integrating factor for this equation is $\mu(t) = exp \left(\int a \, dt \right) = e^{at}$. Multiplying the ODE by $\mu(t)$, we get*

$$e^{at} y' + e^{at} ay = be^{at}$$
$$(e^{at} y)' = be^{at}$$
$$e^{at} y = \int be^{at} dt = \frac{b}{a} e^{at} + c$$
$$y = \frac{b}{a} + ce^{-at}.$$

Since $a > 0$, we have $\lim_{t \to \infty} ce^{-at} = 0$ for any value of c. Therefore $\lim_{t \to \infty} y(t) = \frac{b}{a}$, and this is independent of which solution we pick.

Say $a > 0, c, d$ are constants. A function of the form $y(t) = d + ce^{at}$ is said to be **growing exponentially** and a function of the form $y(t) = d + ce^{-at}$ is said to be **decaying exponentially**. Exponentially growing and decaying functions appear frequently in nature, essentially because they are solutions of a very simple differential equation. An exponentially growing function is clearly unbounded for $t > 0$. On the other hand, an exponentially decaying function has a horizontal asymptote as $t \to \infty$.

We will study some applications involving exponential growth or decay in the next lecture.