MATH 219

Fall 2020

Lecture 1

Lecture notes by Özgür Kişisel

Content: Introduction, Direction Fields, Separable Equations (parts of sections 1.1 and 1.3, section 2.2 (homogenous equations as described in problem 30 is also covered)).

Suggested Problems: (Boyce, DiPrima, 9th edition)

§1.1: 4, 10, 15-20

§1.3: 3, 4, 11, 18, 26

§2.2: 4, 6, 17, 23, 32, 36

1 Introduction

A **differential equation** is a functional equation that contains derivatives and algebraic operations. Such an equation always has one or more **dependent variables** which are functions of one or more **independent variables**. Alternatively we may have several equations containing several variables interrelated with each other; if this is the case we say that we have a **system** of differential equations. It is important to note that the solutions of a differential equation are functions, and not numbers.

Example 1.1 *Say x is a function of t. Consider the differential equation*

$$\frac{dx}{dt} + 5x = e^t.$$

The equation says that the derivative of x with respect to t plus 5 times x should be equal to e^t. The function $x(t)$ is the unknown here and the aim is to find all functions $x(t)$ that satisfy this equation. We will not solve the equation for now. Is it clear to you whether this equation has a solution or not? How many solutions does it have?

Example 1.2 *Say x and y are both functions of t. Consider the system of differential equations*

$$\begin{aligned}\frac{dx}{dt} &+ \quad y = t \\ \frac{dy}{dt} &+ \quad xy = \sin(t)\end{aligned}$$

*In this case, a solution is a **pair** of functions $(x(t), y(t))$.*

Example 1.3 *This time, suppose that u is a function of both x and y, but assume that x and y are two independent variables. In other words we are free to set both x and y to any values we like. Since u is a multivariable function, it makes more sense to consider partial derivatives of u: Expressions like $\frac{du}{dx}$ don't make much sense but $\frac{\partial u}{\partial x}$ or $\frac{\partial u}{\partial y}$ do. For example, let us look at the equation*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x + y.$$

This is a single differential equation that relates the partial derivatives of u with respect to x and y. Solutions will be functions $u(x, y)$ that satisfy this equality. Can you find any?

Definition 1.1 *A differential equation in which there is only one independent variable is called an **ordinary differential equation** (abbreviated to **ODE**). A differential equation in which there is more than one independent variable (and therefore includes partial derivatives) is called a **partial differential equation** (abbreviated to **PDE**).*

Example 1.4 *Let x be a function of t. Consider the differential equation*

$$\frac{d^2x}{dt^2} + x\frac{dx}{dt} + x^2 = 1.$$

This time not only the first derivative, but also the second derivative of x appears in the equation. The equation constrains the function $x(t)$ by relating its second and first derivatives and the function itself.

Definition 1.2 *The highest derivative appearing in a differential equation is called the* ***order*** *of that equation.*

Example 1.5 *The equation* $\dfrac{d^2x}{dt^2} + x\dfrac{dx}{dt} + x^2 = 1$ *has order* 2*. Example 1.1 above is a first order ODE, whereas example 1.2 is a system of first order ODE's and example 1.3 is a first order PDE. The equation*

$$\left(\frac{dx}{dt}\right)^5 + \frac{d^3x}{dx^3} + x = 0$$

has order 3 *(not* 5*: The* 5^{th} *power appearing in the equation is an algebraic operation, so it does not affect its order).*

2 Why are differential equations interesting?

It is a trivial matter to write down as many differential equations as one likes. However, this is not the main focus of the subject; the topic is interesting mainly because of its applications to real life problems. The dependent and independent variables often represent certain physical quantities such as volume, price, velocity, length, time etc. Applications are very diverse, spanning essentially all branches of science and technology.

But why are differential equations useful in such areas? Why do we look at rates of change, or rates of change of rates of change? One possible answer to this question is related to complexity of expressions. Often, understanding or testing how a function **changes** with respect to an independent variable is easier than writing a direct formula for the function in terms of the variable. Actually, the process is often completely opposite to creating a formula for your function by some wizardry. Here is (arguably) the main idea of differential equations: Say we want to write down a formula for $y(t)$. If we know $y(t_0)$ for a certain value of t_0 of t, and if we also know **how fast** $y(t)$ changes when we change t (but now, for all values of t), not necessarily as a function of t only but as a function of both t and y, then maybe we can reconstruct $y(t)$ from this information. If we know the rate of change of $y(t)$ in terms of t only, this is calculus (integration). If we know it in terms of both t and y, it is differential equations.

Can we solve all differential equations by some general algorithms? The answer to this question is an immediate "certainly not". Even, fairly innocent looking differential equations are extremely hard or impossible to solve. We need the differential equation to be simple in some sense if we expect to solve it explicitly. However, simple things tend to occur often in nature, or tend to attract our attention, or they tend to be the things that we work on and design things about. Therefore solving simple differential equations is not a useless task at all.

Another approach to understanding differential equations is to obtain partial information about the solutions even if we cannot come up with explicit formulas. Along with various very interesting theoretical results about general structure of solutions (for example, some nice and useful inequalities rather than equations), there are many interesting and succesful approaches to finding **numerical approximations** to the solutions. These two streams of activities usually take up most of the current research on differential equations. We will not be able to cover these topics here, however this course will provide you with the necessary background to proceed further in these directions.

3 Solutions of first order ODE's

Let us now concentrate on some simple differential equations. For this reason, let us consider a single first order ODE. Therefore we have one dependent variable (say y) and one independent variable (say t) related by a differential equation. A function $y(t)$ that satisfies the equation at all points t in an open interval (a, b) will be called a **solution** of this ODE. Note that, $y(t)$ must be differentiable (and therefore continuous) in order for the equation to make sense. The number a can be any finite number or $-\infty$, similarly the number b can be any finite number greater than a or $+\infty$.

Recall from calculus that a very effective way to understand the behaviour of a function $y(t)$ is by sketching its graph. The same is true for solutions of an ODE. We call such a graph a **solution curve** for the ODE.

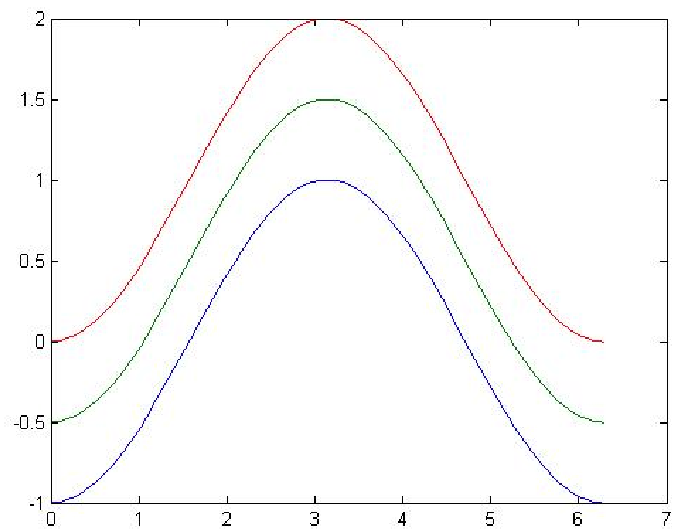
Example 3.1 *An ODE may have many solutions. For example, consider the equation*

$$\frac{dy}{dt} = \sin(t).$$

This is an extremely simple differential equation since on the left hand side we have a first derivative and on the right hand side we have a function of t only - there are no y's that complicate things. This is actually a calculus question. In order to find $y(t)$, integrate the right hand side.

$$y(t) = \int \sin(t)dt = -\cos(t) + c.$$

These are all solutions of this differential equation. Because of the "+c" term, there are infinitely many solutions, depending on c. If we try to graph some of these solutions, we see that any two graphs differ by a constant shift in the vertical direction. The graph below shows three solution curves for three different values of c (and t belonging to the interval $(0, 2\pi)$. Remember that 2π is slightly greater than 6).

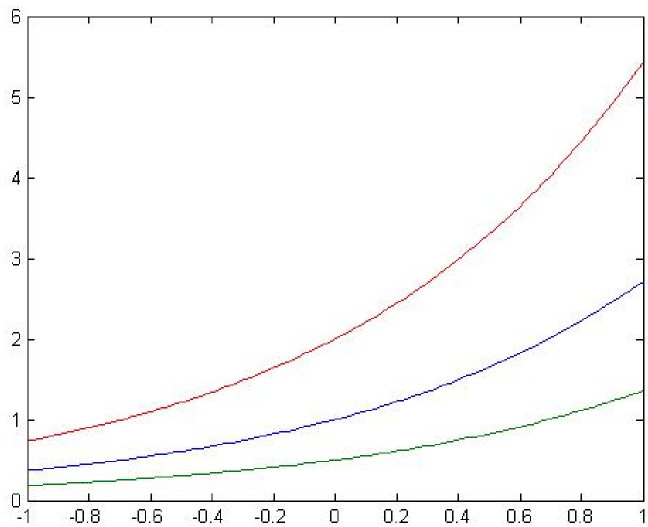


Example 3.2 *As a second example, let us consider the differential equation*

$$\frac{dy}{dt} = y.$$

We cannot solve this example directly with the method used in the last example: If we try to integrate the right hand side with respect to t, we have some function y

that we don't know, so we do not know how to integrate it with respect to t. So we get stuck at this step. We will soon see how to find all solutions of this equation systematically. For now just notice that the equation asks for a function which is equal to its own derivative. We know such a function from calculus: $y(t) = e^t$. This is one solution of the equation, but it is not the only possibility: If we multiply this function by a constant, it is still a solution. So $y(t) = ce^t$ is a solution for any value of the constant c. We again have a family of infinitely many solutions depending on c, however, now the constant c appears multiplicatively (in the previous example it was additive). Let us graph a few solutions again. The ratio of any two solutions is a constant, but their difference is not.



Both of the examples above had infinitely many solutions. If an additional condition on $y(t)$ were given, for instance $y(0) = 1$, then the solution would be unique. In the first example this condition says

$$y(0) = -\cos(0) + c = 1,$$

therefore c must be equal to 2. So there is a unique solution $y(t) = -\cos(t) + 2$. On the graph, this means that we are picking out the curve passing through the point $t = 0$, $y = 1$.

Definition 3.1 *Say we have a first order ODE with dependent variable y and independent variable t. A condition of the form $y(t_0) = y_0$ is called an* ***initial condition****.*

4 Direction Fields

As mentioned before, it is often very difficult to find explicit solutions of an arbitrarily given differential equation. This is true even in the case of first order ODE's. For instance, it is safe to bet that nobody on earth will ever be able to find formulas for explicit solutions of a crazy looking equation like

$$\frac{dy}{dt} = y^3 + y\cos(y) + \sin(t^3) + \frac{1}{ty - \sqrt{y}}.$$

Our aim in this section will *not* be to solve equations, but rather to say something about the solutions without actually finding them. Suppose that we isolate the derivative term in a first order ODE and write the equation in the form

$$\frac{dy}{dt} = f(t, y)$$

where f is some function. Solving the equation in a sense is equivalent to finding the solution curves. We expect to have infinitely many solution curves, depending on a constant c in some way or another. We do not yet know what these curves are, but we have the following geometric information about them: At each point (t, y) on the plane, the value of $f(t, y)$ equals the **slope of the tangent line** to the solution curve passing through this point, simply because it is equal to the value of the derivative. If we had the solution curves, we could easily draw these tangent lines. But the situation is just the opposite: We know what the tangent lines look like and we want to reconstruct the solution curves. So we can do the following:

- First, plot line segments having these slopes ($f(t, y)$ at (t, y)) for as many points (t, y) as possible. Of course, since there are infinitely many points on the plane we cannot plot them all, but by choosing a fine mesh we can draw a large number of them. It is important to notice that we do not need to solve the ODE for this purpose but just need to tabulate the values of $f(t, y)$ at these points.

- Then, sketch approximate solution curves of the differential equation. The key point is that they must be tangent to the line segments constructed above at each of their points. Again, this cannot be accomplished fully, but we can do it approximately in a satisfactory way.

This will give a rough idea about what the solution curves of the differential equation look like. Notice that at the end of this process we will not have formulas for the solution curves, nevertheless we will obtain their approximate graphs. The initial picture that we get after drawing the line segments is called a **direction field** (or a slope field).

There is a practical method which sometimes makes the plotting of the direction field easier: If the curves of the form $f(t, y) = c$ are easy to draw, then we can place them on the plot as a preliminary step (in a way to be erased later on). Along such a curve, the slope of the tangent line will be equal to c throughout, so all of the slope lines will be parallel to each other along the curve. This observation significantly reduces the effort necessary to plot the direction field.

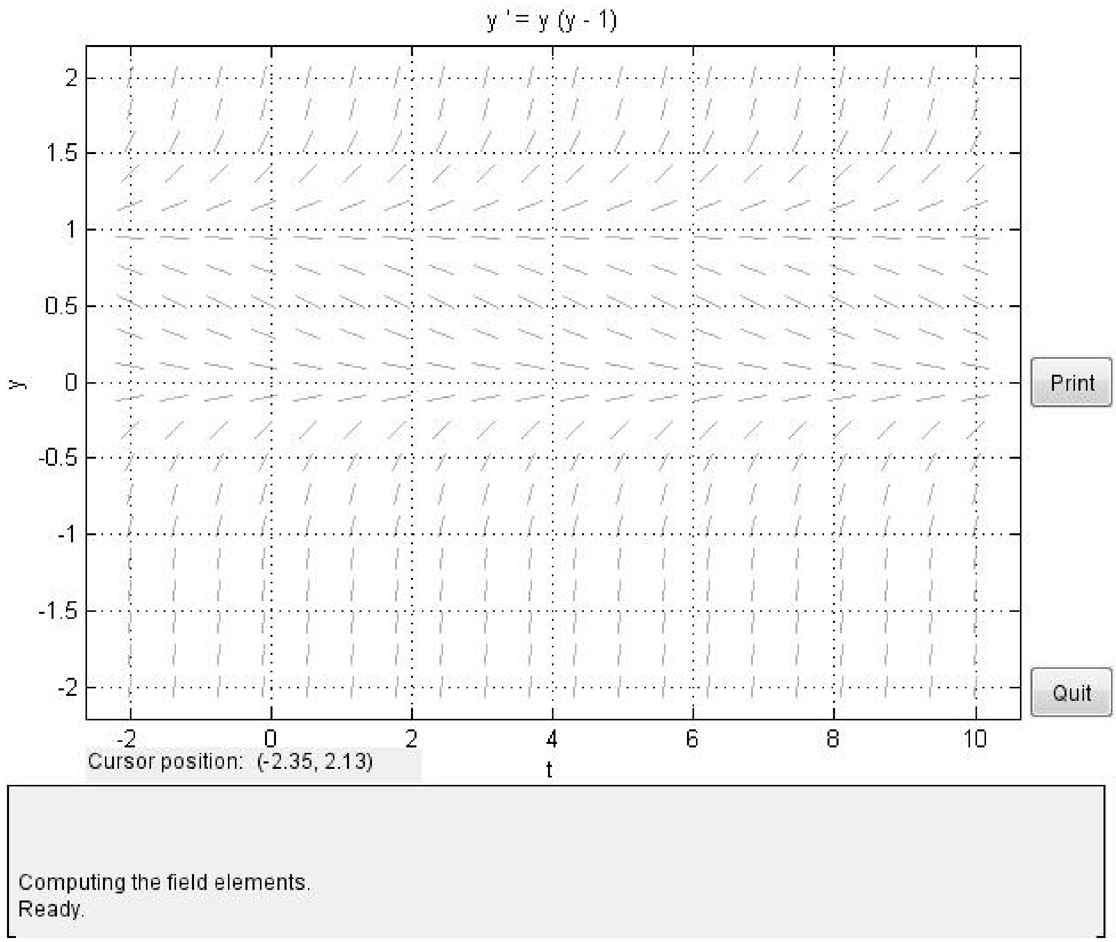
Example 4.1 *Consider the differential equation*

$$\frac{dy}{dt} = y(y-1).$$

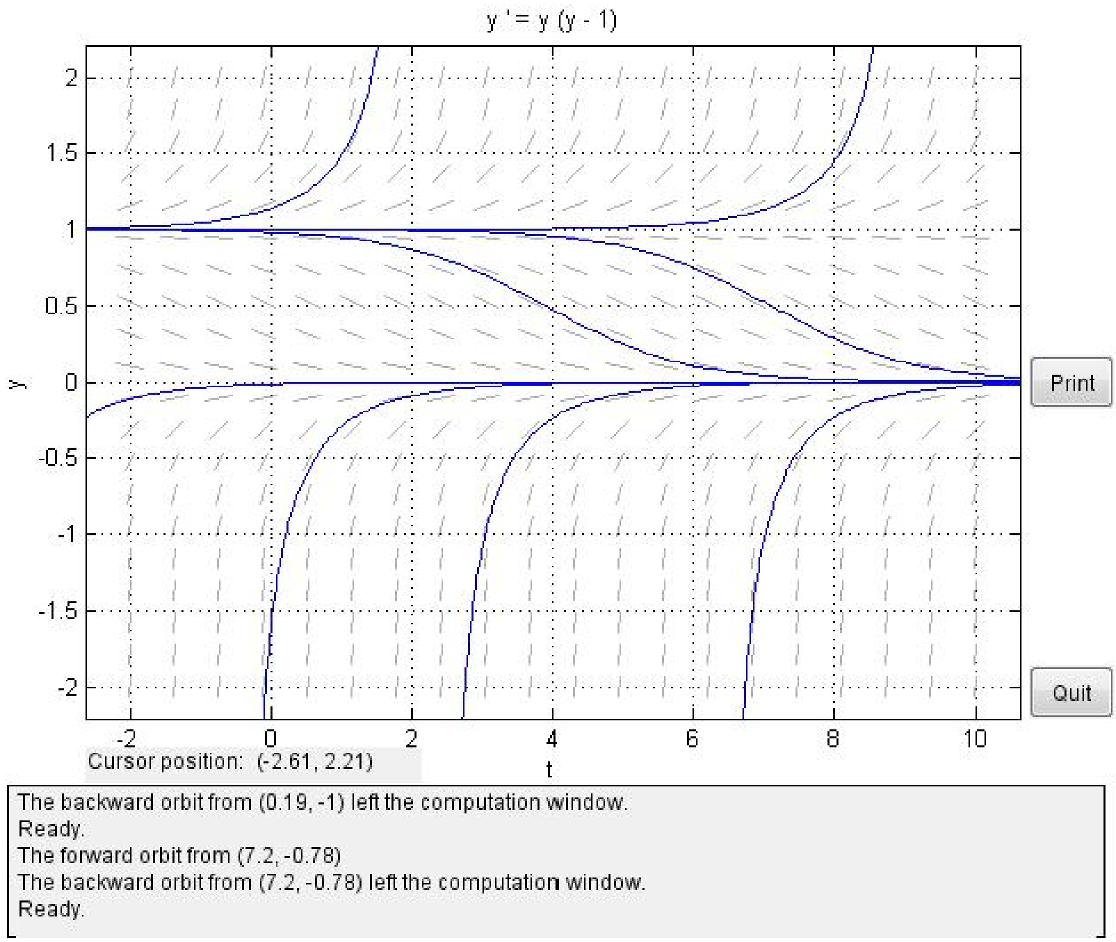
Here, $f(t, y) = y(y-1)$ and it is special in the sense that it depends only on y but not t. Let us find $f(t, y)$ for a few values of y, listed in the table below:

y	*f(t,y)*
-0.5	*0.75*
0	*0*
0.5	*-0.25*
1	*0*
1.5	*0.75*

This means that on the direction field, for any point with y-coordinate -0.5, we should place a line segment of slope 0.75 etc. Expand the table by making this calculation for more values of y, maybe by using a computer. Plotting the line segments with the calculated slopes results in the direction field below:



This plot was actually obtained using the subroutine "dfield8" on Matlab, but it can easily be drawn by hand, given enough time. Now let us plot a few solution curves on the same graph; the rule is: "follow the direction field". The solution curves should be sketched in a way that they are tangent to the line segments at each point.



Using this sketch, we can say several things about the solution curves even though we don't have a single formula for any solution curve at this point:

- *Solutions that have an initial value $y(0)$ between 0 and 1 remain between 0 and 1 and tend to 0 as $t \to \infty$. These solutions are monotone decreasing.*

- *Solutions that have an initial value $y(0)$ larger than 1 tend to infinity (actually even before t has a chance to march off to infinity, but this is more difficult to observe here. We will discuss this later).These solutions are monotone increasing.*

- *Solutions that have an initial value $y(0)$ less than 0 tend to 0 when $t \to \infty$. These solutions are also monotone increasing.*

Of course, these are observations rather than proofs and care must be taken before they are seriously used. For instance, after this observation, one can go back and carefully prove that the solutions are monotone increasing for $y > 1$ by looking at the sign of $y(y-1)$.

5 Separable Equations

Now, let us explicitly solve some equations. In this section our aim is to solve the "simplest" ones in a sense. Believe it or not, you will come across such simple equations as the ones discussed here very often.

As a first attempt for solving a first order ODE, it is enticing to try integrating both sides of the equation. But this method dramatically fails for a general equation of the form $\frac{dy}{dt} = f(t,y)$: If $f(t,y)$ is going to be integrated with respect to t, we need to know what y is in terms of t. But this requires the knowledge of the solution of the problem itself, so there is a circular reasoning here and it doesn't work. Separable equations is a special case where we can somehow algebraically separate the two variables, so that we can do something:

Definition 5.1 *Suppose that $f(t,y)$ can be written in the form $\frac{M(t)}{N(y)}$ for some functions M and N. Then the differential equation $\frac{dy}{dt} = \frac{M(t)}{N(y)}$ is called a **separable** differential equation.*

If our equation is separable then we can find all of its solutions as follows:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{M(t)}{N(y)} \\
N(y)\frac{dy}{dt} &= M(t) \\
\int N(y)\frac{dy}{dt}dt &= \int M(t)dt \\
\int N(y)dy &= \int M(t)dt
\end{aligned}$$

(We can pass to the fourth equation from the third one by using the chain rule.) In the last line, the left hand integral is purely with respect to y and the right hand integral is purely with respect to t. Therefore, in principle both can be evaluated and we obtain a relation between y and t. This is an implicit relation. In some cases we can explicitly solve for $y(t)$, but in general an implicit relation is all that we can find.

It is a fact that in many examples the integrals obtained in this process might be difficult to find, and even impossible sometimes. This we regard as some other type of difficulty irrelevant to our discussion. In the worst case scenario, one can evaluate the integrals by symbolic or numerical integrators; it is still not that bad.

Example 5.1 *Find all solutions of the ODE*

$$\frac{dy}{dt} = te^{t^2-\ln(y^2)}.$$

Solution: *The equation is scary looking at first sight, but with a small algebraic manipulation we can rewrite it as*

$$\frac{dy}{dt} = \frac{te^{t^2}}{y^2}$$

therefore it is separable. Multiply both sides by y^2 and integrate.

$$\int y^2 dy = \int te^{t^2} dt$$
$$\frac{y^3}{3} = \frac{e^{t^2}}{2} + c$$
$$y = \left(\frac{e^{t^2}}{2} + c\right)^{\frac{1}{3}}$$

There are a few things to note here: The integral on the right hand side can be obtained by using the substitution $u = t^2$. Important: There are two indefinite integrals, yet one c is enough since equality of the integrands implies that the antiderivatives differ by a constant. Also note that the constant c appears in the final result in an awkward place; it is neither additive nor multiplicative. Just let it stay where it is.

Example 5.2 *Find all solutions of the ODE*

$$\frac{dy}{dt} = y(y-2)t.$$

Solution: *We may rewrite the equation as*

$$\frac{dy}{dt} = \frac{t}{1/y(y-2)}$$

but some care is needed. The two equations are equivalent if and only if y is different from 0 or 2. We will take care of these cases separately. Now, for a moment assume *that* $y \neq 0$ *and* $y \neq 2$*. Then, as before,*

$$\begin{aligned}\int \frac{dy}{y(y-2)} &= \int t dt\\ \int \frac{dy}{2(y-2)} - \int \frac{dy}{2y} &= \int t dt\\ \frac{1}{2}(\ln|y-2| - \ln|y|) &= \frac{t^2}{2} + c\\ \ln\left|\frac{y-2}{y}\right| &= t^2 + c\\ \frac{y-2}{y} &= ce^{t^2}\\ y &= \frac{2}{1-ce^{t^2}}\end{aligned}$$

A few notes are in order: To pass from the first line to the second line, write $\frac{1}{y(y-2)} = \frac{A}{y-2} + \frac{B}{y}$ *and solve for* A, B*. A second issue: The reader is probably annoyed by the careless use of the constant* c*: Passing from line 3 to line 4, the constant* c *is multiplied by* 2*, but it still takes all possible real values, therefore the new constant can be written as* c *instead of* $2c$*. Passing from line* 4 *to line* 5*, everything is exponentiated, so we should have written* e^c *instead. Lifting the absolute values gives* $\pm e^c$*. This is again an arbitrary constant, it can take all real values except for* 0*. So we should note that in the last equation* c *is an arbitrary constant different from* 0*.*

What about $y = 2$ *and* $y = 0$*? These constant functions are both solutions because* $\frac{d2}{dt} = 2(2-2)t$ *and* $\frac{d0}{dt} = 0(0-2)t$ *are correct identities. Actually, the solution* $y = 2$ *corresponds to the missing value* $c = 0$ *in the formula above. However, the solution* $y = 0$ *does not correspond to any special value of* c*, unless one prefers to think about it as the degenerate case* $c = \infty$*.*

There is a subtle point about the argument concerning $y = 2$ *or* $y = 0$*. You may rightfully ask "if* $y(t) \neq 2$ *for some value* t *could it not be equal to* 2 *for some other value* t*?" After all,* y *is a function, it is not a number. This possibility seems to create more cases which are not discussed in the argument above. The answer (which I think would be unclear for you at this stage) is that everything is alright since there is a fact saying that two different solution curves cannot intersect at a point.*

However this fact depends on a deep and important result, the existence-uniqueness theorem which we will talk about later.

6 Homogenous Equations

Some first order ODE's are not separable, yet they become separable after a simple substitution. In this section we will discuss one general class of such examples: homogenous equations.

Definition 6.1 *A first order ODE is said to be* ***homogenous*** *if it can be written in the form*

$$\frac{dy}{dt} = h\left(\frac{y}{t}\right)$$

for some function h.

The given form of an equation may be deceptive and this can make it tricky to check whether the equation is homogenous. A quick test for homogeneity is as follows: If $\dfrac{dy}{dt} = f(t,y) = h\left(\dfrac{y}{t}\right)$, then $f(\lambda t, \lambda y) = h\left(\frac{\lambda y}{\lambda t}\right) = h\left(\frac{y}{t}\right) = f(t,y)$ for any $\lambda \neq 0$. Therefore, if $f(\lambda t, \lambda y) \neq f(t,y)$ even for one value of λ, then the ODE is not homogenous.

Example 6.1 *Let*

$$\frac{dy}{dt} = t + y.$$

Then $f(\lambda t, \lambda y) = \lambda(t+y) \neq t + y$ for almost all values of λ. Therefore this ODE is not homogenous.

Example 6.2 *Let*

$$\frac{dy}{dt} = \frac{t+y}{t-y}.$$

Then, we can rewrite this equation as

$$\frac{dy}{dt} = \frac{1 + \frac{y}{t}}{1 - \frac{y}{t}}.$$

If we set $h(v) = \dfrac{1+v}{1-v}$ *then*

$$\frac{dy}{dt} = h\left(\frac{y}{t}\right)$$

therefore the equation is homogenous.

Suppose now that we have a homogenous equation $\frac{dy}{dt} = h\left(\frac{y}{t}\right)$. Substitute $v = y/t$. The right hand side will clearly be $h(v)$. In order to compute the left hand side, notice that $y = vt$. By the product rule,

$$\frac{dy}{dt} = \frac{d(vt)}{dt} = v + t\frac{dv}{dt}$$

Therefore the equation can be written as

$$\begin{aligned} v + t\frac{dv}{dt} &= h(v) \\ \frac{dv}{dt} &= \frac{h(v) - v}{t} \end{aligned}$$

This final equation is separable. We reduced everything to a case which we know how to handle.

Example 6.3 *Let us solve the equation*

$$\frac{dy}{dt} = \frac{1 + \frac{y}{t}}{1 - \frac{y}{t}}.$$

Setting $h(v) = \dfrac{1+v}{1-v}$ *as above, we obtain*

$$\begin{aligned} \frac{dv}{dt} &= \frac{h(v) - v}{t} \\ \int \frac{dv}{h(v) - v} &= \int \frac{dt}{t} \\ \int \frac{dv}{\frac{1+v}{1-v} - v} &= \int \frac{dt}{t} \\ \int \frac{(1-v)dv}{1+v^2} &= \ln|t| + c \\ \arctan(v) - \frac{1}{2}\ln|1+v^2| &= \ln|t| + c. \end{aligned}$$

Finally, plugging in $v = \dfrac{y}{t}$ *gives us an implicit relation between* y *and* t*. It seems to me that it should be extremely difficult to write* y *in terms of* t *alone in this problem. Let us leave it like this.*