M E T U Department of Mathematics

| Math 219 Introduction to Differential Equations Fall 2019 Midterm 1 9 November 2019 13:30 | | |
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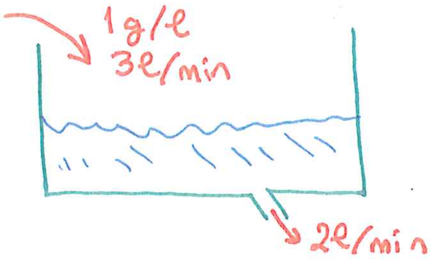
| FULL NAME | STUDENT ID | DURATION 120 MINUTES |
| 5 QUESTIONS ON 4 PAGES | SHOW ALL YOUR WORK | TOTAL 100 POINTS |

*(25 pts)* **1.** A tank with a capacity of $500\ell$ originally contains $200\ell$ of water with $100g$ of salt in solution. Water containing $1g$ of salt per liter is entering at a rate of $3\ell/min$, and the mixture is allowed to flow out of the tank at a rate of $2\ell/min$.

(a) Find the amount of salt in the tank prior to the instant when the solution begins to overflow.




$V(t)$: Volume at time $t$

$\boxed{V(t) = 200 + t \text{ liters}}$

$Q(t)$: Amount of salt in tank at time $t$ (in grams)

$$\boxed{\frac{dQ}{dt} = \text{rate in} - \text{rate out} = 3 - \frac{2Q}{200+t} \quad (g/\ell)}$$

$$\frac{dQ}{dt} + \frac{2Q}{200+t} = 3 \quad \text{(first order linear)}$$

$$\mu(t) = e^{\int 2/200+t\, dt} = e^{2\ln(200+t)} = (200+t)^2$$

$$((200+t)^2 Q)' = 3\cdot(200+t)^2 \Longrightarrow (200+t)^2 \cdot Q = (200+t)^3 + c$$

$$\boxed{Q(t) = (200+t) + \frac{c}{(200+t)^2}}$$

$$Q(0) = 100 \Longrightarrow 100 = 200 + \frac{c}{200^2} \Rightarrow \boxed{c = -100\cdot 200^2 = -4\cdot 10^6}$$

$$\boxed{Q(t) = (200+t) - \frac{4\cdot 10^6}{(200+t)^2}}$$

(b) Find the concentration (in grams per liter) of salt in the tank when it is at the point of overflowing.

The tank begins to overflow at $\boxed{t = 300 \text{ min}}$

$C(t)$: concentration at time $t$.

$$\boxed{C(t) = Q(t)/V(t)} = 1 - \frac{4\cdot 10^6}{(200+t)^3}$$

$$\boxed{C(300) = 1 - \frac{4\cdot 10^6}{(500)^3} = 1 - \frac{4}{125} = 0.968 \ g/\ell}$$

(c) Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

$$\lim_{t\to+\infty} C(t) = \lim_{t\to+\infty} 1 - \frac{4\cdot 10^6}{(200+t)^3} \boxed{= 1}$$

The concentration in part (b) is close to, but less than this limiting capacity.

*(15 pts)* **2.** Consider $(y-xy^3)dx + xdy = 0$, $y(1) = 1$. Find an integrating factor of the form $\mu(xy)$ and solve this initial value problem.

$\mu(xy)\cdot(y-xy^3)\,dx + \mu(xy)\cdot x\,dy = 0$ should be exact.

By the test for exactness

$$\frac{\partial}{\partial y}\left(\mu(xy)\cdot(y-xy^3)\right) = \frac{\partial}{\partial x}\left(\mu(xy)\cdot x\right)$$

$$x\cdot\mu'(xy)\cdot(y-xy^3)+\mu(xy)\cdot(1-3xy^2) = y\cdot\mu'(xy)\cdot x + \mu(xy)$$

$$\mu'\cdot(\cancel{xy}-x^2y^3-\cancel{xy}) = \mu\cdot(\cancel{1}-(\cancel{1}-3xy^2))$$

$$\frac{\mu'}{\mu} = \frac{3xy^2}{-x^2y^3} = -\frac{3}{xy}$$ only depends on $x\cdot y$, so finding such $\mu$ is possible

$$\int\frac{\mu'}{\mu} = \int\frac{-3}{xy} \Rightarrow \ln\mu = -3\ln(xy) \Rightarrow \boxed{\mu(x\cdot y) = (x\cdot y)^{-3}}$$

$$(xy)^{-3}(y-xy^3)\,dx + (xy)^{-3}\cdot x\,dy = 0$$

$$\frac{\partial F}{\partial y} = (xy)^{-3}\cdot x = x^{-2}y^{-3}$$

$$F = \frac{-x^{-2}y^{-2}}{2} + f(x)$$

$$\frac{\partial F}{\partial x} = x^{-3}y^{-2} + f'(x) = (xy)^{-3}(y-xy^3) = x^{-3}y^{-2} - x^{-2}$$

$$f'(x) = -x^{-2} \Rightarrow f(x) = x^{-1}$$

$$\boxed{F(x,y) = \frac{-x^{-2}y^{-2}}{2} + x^{-1} = C}$$

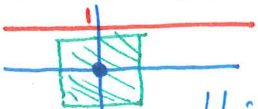
$$(y(1) = 1 \Rightarrow -\frac{1}{2} + 1 = C$$

$$\boxed{\frac{-x^{-2}y^{-2}}{2} + x^{-1} = \frac{1}{2}}$$

*(15 pts)* **3.** $\frac{dy}{dx} = \frac{x^2}{1-y}$, $y(0) = 0$.

(a) Show, without solving, that this initial value problem has a unique solution on some interval $(-h, h)$.

Let $f(x,y) = \frac{x^2}{1-y}$. $\frac{\partial f}{\partial y} = \frac{x^2}{(1-y)^2}$. Both $f$ and $\frac{\partial f}{\partial y}$ are continous away from the line $y=1$. There exists an open rectangle containing $(0,0)$ on which both $f$ and $\frac{\partial f}{\partial y}$ are continuous. Therefore, by the existence-uniqueness thm, this IVP has a unique soln. on some interval $(-h,h)$ for some $h>0$.



(b) Solve the initial value problem and find the largest possible value of $h$.

The equation is separable.

$$\int(1-y)\,dy = \int x^2\,dx$$

$$y - \frac{y^2}{2} = \frac{x^3}{3} + c$$

$$y(0) = 0 \Rightarrow 0 - 0 = 0 + c \Rightarrow \boxed{c = 0}$$

$$\boxed{y^2 - 2y + \frac{2x^3}{3} = 0}$$

$$y = \frac{2 \mp \sqrt{4 - \frac{8x^3}{3}}}{2} = \boxed{1 \mp \sqrt{1 - \frac{2x^3}{3}}}$$

Since $y(0) = 0$, the negative sign must be chosen.

$$\boxed{y = 1 - \sqrt{1 - \frac{2x^3}{3}}}$$

Solution is defined for

$$1 - \frac{2x^3}{3} > 0$$

$$1 > \frac{2x^3}{3} \Rightarrow x^3 < \frac{3}{2}$$

$$\Rightarrow x < \left(\frac{3}{2}\right)^{1/3}$$

So the maximum possible $h$ is

$$\boxed{h = \left(\frac{3}{2}\right)^{1/3}}$$

*(20 pts)* 4. Consider the system

$$\mathrm{x}' = \underbrace{\begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}}_{A} \mathrm{x}$$

(a) Find a fundamental matrix for this system.

This is a constant coefficient, linear, homogeneous system.

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ -1 & -3-\lambda \end{vmatrix} = (-1-\lambda)(-3-\lambda)+1 = \lambda^2+4\lambda+4 = (\lambda+2)^2$$

So, eigenvalues are $\boxed{\lambda_1 = \lambda_2 = -2}$ (repeated eigenvalues)

Eigenvectors: Solve $(A+2I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \vec{v} = \vec{0} \Rightarrow \boxed{\vec{v} = \begin{bmatrix} -k \\ k \end{bmatrix}, \ k \in \mathbb{R} - \{0\}}$$

Since we can only choose 1 linearly indep. eigenvector, we also need a generalized eigenvector.

Say $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Generalized Eigenvectors: Solve $(A+2I)\vec{w} = \vec{v}$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \vec{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \boxed{\vec{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -k \\ k \end{bmatrix}, \ k \in \mathbb{R}} \quad \text{Say } \vec{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\vec{x}^{(1)} = \vec{v}e^{\lambda t} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}, \quad \vec{x}^{(2)} = \vec{v}te^{\lambda t} + \vec{w}e^{\lambda t} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{-2t}$$

$$\boxed{\Psi = \left[ \vec{x}^{(1)} \ \vdots \ \vec{x}^{(2)} \right] = \begin{bmatrix} -e^{-2t} & -te^{-2t} - e^{-2t} \\ e^{-2t} & te^{-2t} \end{bmatrix}}$$

(b) Find the solution of the system that satisfies the initial condition $\mathrm{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

All solutions of the system are $\boxed{\vec{x} = \Psi \vec{c}}$, where $\vec{c}$ is a constant vector.

$$\vec{x}(0) = \Psi(0)\vec{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} -c_1 - c_2 = 1 \\ \boxed{c_1 = 1} \end{array}\right\} \boxed{c_2 = -2}$$

$$\boxed{\vec{x} = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} - 2\left( \begin{bmatrix} -1 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{-2t} \right)}$$

$$x' = \begin{bmatrix} 1 & a \\ 2 & b \end{bmatrix} x$$

where $a, b \in \mathbb{R}$. Suppose that one of the eigenvalues of the coefficient matrix is $+5i$.

(a) Find $a$ and $b$.

Since $+5i$ is an eigenvalue, its complex conjugate $-5i$ must also be an eigenvalue.

$$\Rightarrow \det(A-\lambda I) = \begin{vmatrix} 1-\lambda & a \\ 2 & b-\lambda \end{vmatrix} = (\lambda-5i)(\lambda+5i) = \lambda^2+25$$

$$(1-\lambda)(b-\lambda)-2a = \lambda^2-(b+1)\lambda+(b-2a) = \lambda^2+25 \Rightarrow \boxed{b=-1}, \boxed{a=-13}$$

(b) Find a fundamental set of real valued solutions for this system.

$A = \begin{bmatrix} 1 & -13 \\ 2 & -1 \end{bmatrix}$. Eigenvalues are $\mp 5i$.

Eigenvectors for $+5i$: Solve $(A-5i\cdot I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 1-5i & -13 \\ 2 & -1-5i \end{bmatrix}\vec{v} = \vec{0} \qquad 2v_1-(1+5i)v_2 = 0, \quad \boxed{\vec{v} = \begin{bmatrix} (1+5i) \\ 2 \end{bmatrix} \cdot k}, \ k \in \mathbb{C}\setminus\{0\}$$

Say $\vec{v} = \begin{bmatrix} 1+5i \\ 2 \end{bmatrix}$. Construct a complex valued solution

$$z = e^{5it}\begin{bmatrix} 1+5i \\ 2 \end{bmatrix} = (\cos(5t)+i\sin(5t))\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}+i\begin{bmatrix} 5 \\ 0 \end{bmatrix}\right)$$

$$= \cos(5t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + i\cos(5t)\begin{bmatrix} 5 \\ 0 \end{bmatrix} + i\sin(5t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \sin(5t)\begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(5t)-5\sin(5t) \\ 2\cos(5t) \end{bmatrix} + i\begin{bmatrix} 5\cos(5t)+\sin(5t) \\ 2\sin(5t) \end{bmatrix}$$

Then, $\bar{z}$ will be another complex valued solution, attached to the eigenvalue $-5i$.

$\vec{x}^{(1)} = \dfrac{z+\bar{z}}{2}$ and $\vec{x}^{(2)} = \dfrac{z-\bar{z}}{2i}$ are real valued solutions.

$$\boxed{\vec{x}^{(1)} = \begin{bmatrix} \cos(5t)-5\sin(5t) \\ 2\cos(5t) \end{bmatrix}, \quad \vec{x}^{(2)} = \begin{bmatrix} 5\cos(5t)+\sin(5t) \\ 2\sin(5t) \end{bmatrix}}$$