M E T U
Department of Mathematics

	Introduction to Differential Equations MidTerm 1	
Code: *Math 219* Semester: *Spring 2019* Date: *6 April 2019* Time: *17:00* Duration: *120 minutes*	Last Name: Name: Department: Student No.: Signature: 4 QUESTIONS ON 4 PAGES TOTAL 100 POINTS	

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Question 1 (25 pts) Consider the differential equation

$$\left(\frac{\sin(y)}{y} - 2e^{-x}\sin(x)\right)dx + \left(\frac{\cos(y) + 2e^{-x}\cos(x)}{y}\right)dy = 0.$$

(a) Show that $\mu(x,y) = ye^x$ is an integrating factor by using the test for exactness.

Multiply with $\mu(x,y)$.

$$\underbrace{(e^x\sin(y) - 2y\sin(x))}_{M}dx + \underbrace{(e^x\cos(y) + 2\cos(x))}_{N}dy = 0 \quad (*)$$

$$\frac{\partial M}{\partial y} = e^x\cos(y) - 2\sin(x) = \frac{\partial N}{\partial x} = e^x\cos(y) - 2\sin(x)$$

Since equality holds on all of $\mathbb{R}^2$ and $\mathbb{R}^2$ is simply connected, $(*)$ is exact. Therefore $\mu(x,y)$ is an integrating factor.

(b) Find all solutions of the equation.

$$\underbrace{(e^x\sin(y) - 2y\sin(x))}_{M}dx + \underbrace{(e^x\cos(y) + 2\cos(x))}_{N}dy = 0$$

Find a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$

$$\frac{\partial F}{\partial x} = M \Rightarrow \cancel{F=} e^x\sin(y) + 2y\cos(x) + f(y)$$

$$\Rightarrow \frac{\partial F}{\partial y} = e^x\cos(y) + 2\cos(x) + f'(y).$$ If we take $f=0$ then $\frac{\partial F}{\partial y} = N$ holds.

$$F(x,y) = e^x\sin(y) + 2y\cos(x)$$

Solutions of the ODE are

$$\boxed{e^x\sin(y) + 2y\cos(x) = c}$$ where $c \in \mathbb{R}$ is a constant.

(c) Find the solution that satisfies $y(\pi) = \pi$.

$$e^\pi \underbrace{\sin(\pi)}_{0} + 2\cdot\pi\underbrace{\cos(\pi)}_{-1} = c \Rightarrow c = -2\pi$$

$$\boxed{e^x\sin(y) + 2y\cos(x) = -2\pi}$$

Question 2 (25 pts) Consider the differential equation

$$\frac{ds}{dx} = 2x + 4x^3 e^{-s}.$$

(a) Make the substitution $y = e^s$ and show that the resulting equation is linear in y.

$y = e^s \Rightarrow \frac{dy}{dx} = e^s \frac{ds}{dx} = y \frac{ds}{dx}$. Rewrite the ODE

$\frac{1}{y}\frac{dy}{dx} = 2x + 4x^3 \cdot \frac{1}{y}$

$\frac{dy}{dx} - 2x \cdot y = 4x^3$. The equation is of the form $\frac{dy}{dx} + p(x)\cdot y = q(x)$, so it is linear. $\left(\begin{array}{l} p(x) = -2x \\ q(x) = 4x^3 \end{array}\right)$

(b) Find the solution of the equation that satisfies $s(0) = \ln(2)$.

Use an integrating factor: $\mu(x) = e^{\int p(x)\,dx} = e^{\int -2x\,dx} = e^{-x^2}$

$(e^{-x^2} \cdot y)' = 4x^3 e^{-x^2}$

$e^{-x^2} \cdot y = \int 4x^3 e^{-x^2}\,dx$. Let $z = x^2$. Then $dz = 2x\,dx$

$= \int 2z e^{-z}\,dz$. Apply int. by parts with

$\begin{array}{|l} u = 2z,\ dv = e^{-z}\,dz \\ du = 2\,dz,\ v = -e^{-z} \end{array}$

$= 2z \cdot (-e^{-z}) - \int -e^{-z} \cdot 2\,dz$

$= -2z \cdot e^{-z} - 2e^{-z} + c$

$= -2x^2 e^{-x^2} - 2e^{-x^2} + c$

$\Rightarrow y = -2x^2 - 2 + ce^{x^2}$

$e^s = -2x^2 - 2 + ce^{x^2} \Rightarrow s = \ln(-2x^2 - 2 + ce^{x^2})$

$s(0) = \ln 2 \Rightarrow 2 = -2 \cdot 0 - 2 + c \cdot e^0 \Rightarrow c = 4$

$$\boxed{s = \ln(4e^{x^2} - 2x^2 - 2)}$$

(c) Show that the interval $[0, \infty)$ is a subset of the domain of the solution in part (b).

It is enough to show that $4e^{x^2} - 2x^2 - 2 > 0$ for $x \in [0, \infty)$, since then $\ln$ will be defined at the relevant point. Say $f(x) = 4e^{x^2} - 2x^2 - 2$

At $x = 0$: $f(0) = 4 - 2 = 2 > 0$.

$f'(x) = 8xe^{x^2} - 4x = 4x \cdot (2e^{x^2} - 1)$

On $(0, \infty)$, $x > 0$ and $2e^{x^2} - 1 > 0 \Rightarrow f'(x) > 0$.

Since $f(0) > 0$ and $f'(x) > 0$ for all $x \in (0, \infty)$, f is monotone increasing, therefore $f(x) > 0$ on all of $[0, \infty)$

Question 3 (25 pts) Solve the nonhomogeneous system

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 1 & -4 \\ 5 & -3 \end{bmatrix}}_{A} \mathbf{x} + \underbrace{\begin{bmatrix} e^{-t} \\ 2e^{-t} \end{bmatrix}}_{\vec{b}}.$$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -4 \\ 5 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda)+20 = \lambda^2+2\lambda+17$$
$$=(\lambda+1)^2+16$$

Therefore, $\lambda_{1,2} = -1 \mp 4i$

Eigenvectors for $-1+4i$

$$\left[\begin{array}{cc|c} 2-4i & -4 & 0 \\ 5 & -2-4i & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & \frac{-2-4i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 2+4i \\ 5 \end{bmatrix}$ is an eigenvector.

$$\vec{x} = e^{(-1+4i)t}\begin{bmatrix} 2+4i \\ 5 \end{bmatrix} = e^{-t}(\cos(4t)+i\sin(4t))\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix} + i\begin{bmatrix} 4 \\ 0 \end{bmatrix}\right)$$

$$= e^{-t}\left(\underbrace{\begin{bmatrix} 2\cos(4t)-4\sin(4t) \\ 5\cos(4t) \end{bmatrix}}_{e^{t}\vec{x}^{(1)}} + i\underbrace{\begin{bmatrix} 2\sin(4t)+4\cos(4t) \\ 5\sin(4t) \end{bmatrix}}_{e^{t}\vec{x}^{(2)}}\right)$$

$$\Psi = \left[\vec{x}^{(1)} \,\middle|\, \vec{x}^{(2)}\right] = \begin{bmatrix} 2e^{-t}\cos(4t) - 4e^{-t}\sin(4t) & 2e^{-t}\sin(4t)+4e^{-t}\cos(4t) \\ 5e^{-t}\cos(4t) & 5e^{-t}\sin(4t) \end{bmatrix}$$

$$\det\Psi = e^{-2t}\left(\cancel{10\cos(4t)\sin(4t)} - 20\sin^2(4t) - \cancel{10\cos(4t)\sin(4t)} - 20\cos^2(4t)\right)$$
$$= -20e^{-2t}$$

$$\Psi^{-1}(t) = \frac{-1}{20}e^{2t}\begin{bmatrix} 5e^{-t}\sin(4t) & -2e^{-t}\sin(4t)-4e^{-t}\cos(4t) \\ -5e^{-t}\cos(4t) & 2e^{-t}\cos(4t)-4e^{-t}\sin(4t) \end{bmatrix}$$

$$= \frac{-1}{20}\begin{bmatrix} 5e^{t}\sin(4t) & -2e^{t}\sin(4t)-4e^{t}\cos(4t) \\ -5e^{t}\cos(4t) & 2e^{t}\cos(4t)-4e^{t}\sin(4t) \end{bmatrix}$$

$$\Psi^{-1}(t)\begin{bmatrix} e^{-t} \\ 2e^{-t} \end{bmatrix} = \frac{-1}{20}\begin{bmatrix} \sin(4t)-8\cos(4t) \\ -\cos(4t)-8\sin(4t) \end{bmatrix}$$

$$\int \Psi^{-1}\vec{b}\,dt = \frac{-1}{20}\left(\begin{bmatrix} -\frac{\cos(4t)}{4} - 2\sin(4t) \\ -\frac{\sin(4t)}{4} + 2\cos(4t) \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right)$$

$$\vec{x} = \Psi\int\Psi^{-1}\vec{b}\,dt = -\frac{e^{-t}}{20}\begin{bmatrix} 2\cos(4t)-4\sin(4t) & 2\sin(4t)+4\cos(4t) \\ 5\cos(4t) & 5\sin(4t) \end{bmatrix}$$
$$\cdot\left(\begin{bmatrix} -\frac{\cos(4t)}{4} - 2\sin(4t) \\ -\frac{\sin(4t)}{4} + 2\cos(4t) \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right)$$

Question 4 (25 pts) Let $a \in \mathbb{R}$. Consider the system of differential equations

$$\mathbf{x}' = \underbrace{\begin{bmatrix} -2 & 0 & 1 \\ 0 & -3 & 1 \\ 0 & a & -1 \end{bmatrix}}_{A} \mathbf{x}$$

(a) Find the value of a such that the system has a constant solution $\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ different from $\mathbf{0}$. (Do not solve the system.)

$\vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \vec{x}' = \vec{0} = A\vec{x}$. A must have a 0 eigenvalue $\Rightarrow \det A = 0$

$$\begin{vmatrix} -2 & 0 & 1 \\ 0 & -3 & 1 \\ 0 & a & -1 \end{vmatrix} = -2\begin{vmatrix} -3 & 1 \\ a & -1 \end{vmatrix} = -2(3-a) \Rightarrow \boxed{a=3}$$

(b) Find the value of a for which the coefficient matrix has only one eigenvalue. Find all solutions of the system for this value of a.

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 0 & 1 \\ 0 & -3-\lambda & 1 \\ 0 & a & -1-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} -3-\lambda & 1 \\ a & -1-\lambda \end{vmatrix}$$

$$= (-2-\lambda)\left[(-3-\lambda)(-1-\lambda) - a\right] = (-2-\lambda)(\lambda^2 + 4\lambda + 3 - a)$$

$\lambda_1 = -2$. In order to have $\lambda_2 = \lambda_3 = -2$, we must have $\boxed{a=-1}$

eigenvectors of A: Solve $(A+2I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

v_1 free, $v_2 = v_3 = 0$, $\vec{v} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$ $k \neq 0$

One indep. eigenvector $\Rightarrow$ One Jordan block

$$\Rightarrow J = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}. \quad P = \left[\vec{v}^{(1)} \middle| \vec{v}^{(2)} \middle| \vec{v}^{(3)}\right]$$

$$AP = \left[A\vec{v}^{(1)} \middle| A\vec{v}^{(2)} \middle| A\vec{v}^{(3)}\right] = PJ = \left[-2\vec{v}^{(1)} \middle| \vec{v}^{(1)} - 2\vec{v}^{(2)} \middle| \vec{v}^{(2)} - 2\vec{v}^{(3)}\right]$$

$\vec{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $(A+2I)\vec{v}^{(2)} = \vec{v}^{(1)}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right].$$ A solution is $\vec{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$(A+2I)\vec{v}^{(3)} = \vec{v}^{(2)}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right].$$ A soln. is $\vec{v}^{(3)} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ $\Rightarrow P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\Psi = Pe^{Jt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} e^{-2t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} = e^{-2t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t-1 \\ 0 & 1 & t \end{bmatrix}. \quad \vec{x} = \Psi\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$