Name:

Student number:

KEY

1	
2	
3	
4	
5	
Σ	

METU MATH 125, Final Exam

Sunday, January 5, 2020, at 13:30 (100 minutes), totally 80 points

Instructor: E.Solak

Instructions: Please, show clearly the logic of your solutions.

Problem 1. (16 pts.) What must be the values of a, b, c and d so that the row echelon form of the matrix $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & a & b \\ c & d & 8 \end{bmatrix}$ has rank 1? Explain your answer.

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & a & b \\ c & d & 8 \end{bmatrix} \underset{-2R_1+R_2}{\sim} \begin{bmatrix} 1 & 3 & 4 \\ 0 & a-6 & b-8 \\ c & d & 8 \end{bmatrix} \underset{-cR_1+R_3}{\sim}$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & a-6 & b-8 \\ 0 & d-3c & 8-4c \end{bmatrix}$$

So that the row echelon form of A has rank 1,

$a-6=0$, $b-8=0$, $d=3c$ and $8-4c=0$, i.e.,

$\boxed{a=6, \quad b=8, \quad c=2 \quad \text{and} \quad d=6}$

Problem 2. (12+4=16 pts.) Let $f : S \to T$ be a function. Let R be a relation on S defined by

$$s_1 R s_2 \quad \text{if and only if} \quad f(s_1) = f(s_2) \quad \text{for} \quad s_1, s_2 \in S$$

(a) Show that R is an equivalence relation.

We need to show that R is symmetric, reflexive and transitive.

(i) Since $f(s) = f(s)$ for all $s \in S$, $(s,s) \in R$ for all $s \in S$, which implies R is reflexive.

(ii) If $s_1 R s_2$ then $f(s_1) = f(s_2)$. Since $f(s_1) = f(s_2)$ implies $f(s_2) = f(s_1)$, $(s_2, s_1) \in R$, and so R is symmetric.

(iii) If $s_1 R s_2$ and $s_2 R s_3$ then $f(s_1) = f(s_2)$ and $f(s_2) = f(s_3)$.

$f(s_1) = f(s_2)$ and $f(s_2) = f(s_3) \Rightarrow f(s_1) = f(s_2) = f(s_3)$

$\Rightarrow f(s_1) = f(s_3)$

$\Rightarrow s_1 R s_3 \Rightarrow R$ is transitive.

By (i), (ii) and (iii), R is an equivalence relation.

(b) Write $[s]$ for $s \in S$, where $[s]$ denotes the equivalence class of s.

$$[s] = \{ t \in S \mid (s,t) \in R \}$$
$$= \{ t \in S \mid f(s) = f(t) \}$$

Problem 3. **(20 pts.)** Find the general term of the non-homogenous recurrence relation

$$a_{n+2} = -2a_{n+1} + 3a_n + 3^n \quad \text{for} \quad n \geq 0,$$

with the initial conditions $a_0 = \frac{13}{12}$, $a_1 = \frac{15}{12}$ and $a_2 = \frac{21}{12}$.

$$a_{n+3} = -2a_{n+2} + 3a_{n+1} + 3^{n+1}$$

$$-3a_{n+2} = +6a_{n+1} - 9a_n - 3^{n+1}$$

$$a_{n+3} - a_{n+2} - 9a_{n+1} + 9a_n = 0 \qquad (*)$$

$(*)$ is a homogenous recurrence relation and its characteristic equation is;

$$r^3 - r^2 - 9r + 9 = 0 \quad \text{of degree (order) } 3 \qquad (**)$$

$$r(r^2 - 9) - (r^2 - 9) = 0$$

$$(r^2 - 9)(r - 1) = 0$$

$$(r-3)(r+3)(r-1) = 0 \quad \Rightarrow r_1 = 1,\ r_2 = 3 \text{ and } r_3 = -3 \text{ are roots of } (**).$$

Hence the general term of the sequence (a_n) given above is;

$$a_n = c_1 (1)^n + c_2 (3)^n + c_3 (-3)^n$$

$$a_n = c_1 + c_2 3^n + c_3 (-3)^n$$

By using initial conditions, we compute c_1, c_2 and c_3;

$$a_0 = c_1 + c_2 + c_3 = 13/12$$

$$a_1 = c_1 + 3c_2 - 3c_3 = 15/12 \quad \Rightarrow \quad c_1 = 1, \quad c_2 = 1/12, \quad c_3 = 0$$

$$a_2 = c_1 + 9c_2 + 9c_3 = 21/12$$

Hence, $a_n = 1 + \dfrac{1}{12}(3^n)$

Problem 4. (16 pts.) Let (a_n) be a sequence given by a recurrence relation

$$a_n = a_{n-1} + (n+1)! - n!$$

If $a_0 = 0$ then find a_{100}.

$$a_1 = a_0 + \cancel{2!} - 1!$$
$$a_2 = a_1 + \cancel{3!} - \cancel{2!}$$
$$a_3 = a_2 + \cancel{4!} - \cancel{3!}$$
$$\vdots$$
$$a_{99} = a_{98} + \cancel{100!} - \cancel{99!}$$
$$a_{100} = a_{99} + 101! - \cancel{100!}$$

$$\cancel{a_1} + \cancel{a_2} + \dots + \cancel{a_{99}} + a_{100} = a_0 + \cancel{a_1} + \dots + \cancel{a_{99}} + 101! - 1!$$

$$a_{100} = a_0 + 101! - 1!$$

$$a_{100} = 101! - 1! \quad \text{since } a_0 = 0$$

Problem 5. (12 pts.) Write the general term of the sequence whose characteristic equation is

$$(r-3)^3(r^2+1)(r+6) = 0$$

The general term is given as

$$a_n = c_1(3)^n + c_2 n (3)^n + c_3 n^2 (3)^n + c_4 (i)^n + c_5(-i)^n + c_6(-6)^n$$