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Name:

Student number:

METU MATH 125, MIDTERM 1

Saturday, November 9, 2019, at 13:30 (100 minutes), totally 50 points

Instructions: Please, show clearly the logic of your solutions.

Problem 1. (4 pts.) Let $\mathcal{P}(A) = \big\{\varnothing, \{a\}, \{\{a\}\}, \{a, \{a\}\}\big\}$ be the power set of a set A. Find $P(A) \cap A$.

$|P(A)| = 4 \Rightarrow |A| = 2$

$A = \{ a, \{a\} \}$

So, $P(A) \cap A = \{ \{a\} \}$

Problem 2. (8 pts.) Let

$$A = \{x \in \mathbb{R} \mid x^2 - 11x + 30 < 0\}$$
$$B = \{x \in \mathbb{R} \mid 5 < x < 6\}$$

Show that $A = B$.

To show that $A = B$ we need to show that $A \subseteq B$ and $B \subseteq A$.

First show that $A \subseteq B$. Let $x \in A$. Then $x \in \mathbb{R}$ and $x^2 - 11x + 30 < 0$. So, $x^2 - 11x + 30 = (x-5)(x-6) < 0$

Case 1: $x - 5 > 0$ and $x - 6 < 0 \Rightarrow 5 < x < 6 \Rightarrow x \in B$

Case 2: $x - 5 < 0$ and $x - 6 > 0 \Rightarrow x < 5 \wedge x > 6$, not possible.

Hence, $A \subseteq B$

Next show that $B \subseteq A$. Let $x \in B$. Then $x \in \mathbb{R}$ and $5 < x < 6$. So, $x - 5 > 0$ and $x - 6 < 0$. Hence, $(x-5)(x-6) = x^2 - 11x + 30 < 0$ and this implies $x \in A$.

$B \subseteq A$.

Problem 3. **(12=5+5+2 pts.)** Let $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{5\}$ be a function defined by

$$f(x) = \frac{5x+2}{x-3}$$

(a) Prove that $f(x)$ is one-to-one.

Assume $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{R} - \{3\}$. Then by definition of f, $\frac{5x_1+2}{x_1-3} = \frac{5x_2+2}{x_2-3}$.

$$5x_1x_2 - 15x_1 + 2x_2 - 6 = 5x_1x_2 - 15x_2 + 2x_1 - 6$$

$$\Rightarrow -17x_1 = -17x_2 \Rightarrow x_1 = x_2$$

$\Rightarrow f$ is 1-1.

(b) Prove that $f(x)$ is onto.

For any $y \in \mathbb{R} - \{5\}$ we need to show that $\exists x \in \mathbb{R} - \{3\}$ such that $f(x) = y$.

$$y = \frac{5x+2}{x-3} \Rightarrow xy - 3y = 5x+2 \Rightarrow xy - 3y = 5x+2$$

$$\Rightarrow x = \frac{3y+2}{y-5}$$

Put $x = \frac{3y+2}{y-5}$, we get $f(x) = y$.

This show f is onto.

(c) Find $f^{-1}(x)$ for $x \in \mathbb{R} - \{5\}$.

By part (b), it is easy to see that

$$f^{-1}(x) = \frac{3x+2}{x-5} \quad \text{for } x \in \mathbb{R} - \{5\}.$$

Problem 4. **(6 pts.)** Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be a function defined by $g(m, n) = 2n - 4m$.

(a) Is g is one-to-one? Prove or give a counterexample. No!

Since $g(1,2) = g(2,4) = 0$ but $(1,2) \neq (2,4)$, g fails to be 1-1.

(b) Is g onto? Prove or give a counterexample. No!

None of the odd numbers has a pre image in the domain of f. For example, there is no $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ such that $g(m,n) = 1$.

Problem 5. **(8 pts.)** Let R be a relation defined on $\mathbb{N} = \{1, 2, 3, \cdots\}$ by

$$aRb \text{ if and only if } \frac{a}{b} \in \mathbb{Z}$$

Which of the properties reflexive, symmetric, antisymmetric and transitive does the relation R satisfy? Justify your answer.

Since $\frac{a}{a} = 1 \in \mathbb{Z}$ for $\underline{\text{any}}$ $a \in \mathbb{N}$, aRa for all $a \in A$, which shows R is reflexive.

R is not symmetric since for example $\frac{4}{2} = 2 \in \mathbb{Z} \Rightarrow 4R2$ but since $\frac{2}{4} \notin \mathbb{Z}$, ~~a~~ $2 \not R 4$

R is anti-symmetric since:

if aRb and bRa then $\frac{a}{b} = m \in \mathbb{Z}$ and $\frac{b}{a} = n \in \mathbb{Z}$

and $\frac{a}{b} \cdot \frac{b}{a} = mn = 1 \Rightarrow m, n \in \{1\}$ since $a, b \in \mathbb{N}$

$\Rightarrow a = b$

R is transitive since: if aRb and bRc. Then

$\exists$ $\frac{a}{b} = m \wedge \frac{b}{c} = n$ for $a, b, c \in \mathbb{N}$ and $m, n \in \mathbb{Z}$

then $\frac{a}{\cancel{b}} \cdot \frac{\cancel{b}}{c} = mn = \frac{a}{c} \in \mathbb{Z} \Rightarrow aRc$.

Problem 6. (12=9+3 pts.) Let R be a relation on $\mathbb{Z} \times \mathbb{Z}$ defined by

$$(a, b)R(c, d) \text{ if and only if } a + d = b + c$$

(a) Prove that R is an equivalence relation.

Since $a+b = b+b$ for any $(a,b) \in Z$, $(a,b)R(a,b)$ for all $(a,b) \in Z \times Z$, R is reflexive.

If $(a,b)R(c,d)$ for $a,b,c,d \in Z$ then $a+d = b+c$. Since $b+c = c+b = d+a$, $(c,d)R(a,b)$,

R is symmetric.

if $(a,b)R(c,d) \wedge (c,d)R(e,f)$ for $a,b,c,d,e,f \in Z$

then $a+d = b+c$ and $c+f = d+e$.

Since $a+d = b+c \Rightarrow a+d+f = b+c+f$

$\Rightarrow a+\cancel{d}+f = b+\cancel{d}+e \Rightarrow a+f = b+e$

$\Rightarrow (a,b)R(e,f)$

$\Rightarrow$ R is transitive.

(b) Write down three elements of the equivalence class of $(1, 2)$.

$$[(1,2)] = \{(c,d) \mid 1+d = 2+c\}$$
$$= \{(c, 1+c) \mid c \in Z\}$$

$$[(1,2)] = \{(1,2), (2,3), (3,4), (4,5), \ldots\}$$