

M E T U

Department of Mathematics

CALCULUS WITH ANALYTIC GEOMETRY	
MidTerm 1	
Code : <i>Math 119</i> Acad. Year : <i>2015-2016</i> Semester : <i>Fall</i> Coordinator: <i>Muhiddin Uğuz</i> Date : <i>November, 21, 2015</i> Time : <i>9:30</i> Duration : <i>119 minutes</i>	Last Name : _____ Name : _____ Student No. : _____ Department : _____ Section : _____ Signature : _____
6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS	
1 2 3 4 5 6 Muhiddin Uğuz Digitally signed by Muhiddin Uğuz DN: cn=Muhiddin Uğuz, o=METU, ou=Mathematics, email=muhiddin@metu.edu.tr, c=TR Date: 2015.11.21 15:37:45 +0200	SHOW YOUR WORK

(6+6+6+6 pts) 1. Evaluate each of the following limits if it exists. (DO NOT USE L'HOSPITAL'S RULE)

(a) $\lim_{x \rightarrow +\infty} (x - \sqrt{x^2 - 4x + 1})$ $\infty - \infty$ type

$$= \lim_{x \rightarrow +\infty} (x - \sqrt{x^2 - 4x + 1}) \cdot \frac{(x + \sqrt{x^2 - 4x + 1})}{(x + \sqrt{x^2 - 4x + 1})} = \lim_{x \rightarrow +\infty} \frac{x^2 - x^2 + 4x - 1}{x + \sqrt{x^2 - 4x + 1}}$$

$x \rightarrow +\infty$
 $\Rightarrow$ $\lim_{x \rightarrow +\infty} \frac{x(4 - \frac{1}{x})}{x(1 + \sqrt{1 - \frac{4}{x} + \frac{1}{x^2}})} = \frac{4}{1+1} = 2$

(b) $\lim_{x \rightarrow 0} \frac{3^{(2+x)^2} - 81}{x}$

$$= \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = f'(0) \text{ where } f(x) = 3^{(2+x)^2}$$

$$= f'(0) = 3^4 \cdot 2 \cdot 2 \cdot \ln 3 = 324 \ln 3$$

$\frac{d}{dx} \ln f(x) = (2+x)^2 \ln 3$
 $\frac{f'(x)}{f(x)} = 2(2+x) \ln 3$
 $f'(x) = 3^{(2+x)^2} \cdot 2 \cdot (2+x) \ln 3$

(c) $\lim_{x \rightarrow 0} \frac{\sin 5x}{\tan 7x} = \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \cdot \frac{5x}{\frac{\sin 7x}{7x} \cdot 7x} = \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \cdot \lim_{x \rightarrow 0} \frac{5x}{\sin 7x} \cdot \lim_{x \rightarrow 0} \frac{5}{7}$

Since each limit exists $\Rightarrow \frac{5}{7} \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \cdot \lim_{x \rightarrow 0} \frac{5x}{\sin 7x} \cdot \frac{5}{7} = \frac{5}{7} \cdot 1 \cdot 1 \cdot 1 = \frac{5}{7}$

(d) $\lim_{x \rightarrow +\infty} \frac{\sin(x^2 + 1)}{x^2}$

$$-1 \leq \sin(x^2 + 1) \leq 1 \quad \forall x \in \mathbb{R}$$

$$-\frac{1}{x^2} \leq \frac{\sin(x^2 + 1)}{x^2} \leq \frac{1}{x^2} \quad \forall x > 0 \quad \& \quad \lim_{x \rightarrow +\infty} \pm \frac{1}{x^2} = 0$$

Therefore, by squeeze Theorem, $\lim_{x \rightarrow +\infty} \frac{\sin(x^2 + 1)}{x^2} = 0$

(12 pts) 2.

Find an equation of the tangent line to the curve $x^y = y^x$ at the point $(2, 4)$.

$$x^y = y^x \Rightarrow y \ln x = x \ln y \Rightarrow \text{Take derivative of both sides with respect to } x;$$

$$y' \ln x + y \frac{1}{x} = \ln y + x \frac{1}{y} \cdot y'$$

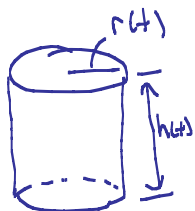
$$\Rightarrow y'(x, y) = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}} \Rightarrow y'(2, 4) = \frac{\ln 4 - \frac{4}{2}}{\ln 2 - \frac{2}{4}} = \frac{4 \ln 2 - 4}{2 \ln 2 - 1}$$

= slope of tangent line to the curve $x^y = y^x$ at $(2, 4)$

Eqn. of tangent line:

$$\frac{y-4}{x-2} = \boxed{\frac{4 \ln 2 - 4}{2 \ln 2 - 1}} = m \Rightarrow y = mx - 2m + 4.$$

(12 pts) 3. A cylinder is expanding so that its volume is increasing at a rate of $100\pi \text{ cm}^3/\text{sec}$. If the radius of the cylinder is increasing at a rate of $\frac{1}{2} \text{ cm/sec}$ when the radius is 5 cm and height is 10 cm , how fast is the **surface area** A of the cylinder changing at that moment?



$$V(t) = r^2(t) \pi h(t) \quad \forall t$$

$$\left. \frac{dV}{dt} \right|_t = 100\pi \text{ cm}^3/\text{sec} \quad \left. \frac{dr}{dt} \right|_{\substack{r(t)=5 \text{ cm} \\ h(t)=10 \text{ cm}}} = \frac{1}{2} \text{ cm/sec}$$

$$A(t) = 2r^2(t)\pi + 2\pi r(t)h(t)$$

$$= 2\pi \left(r^2(t) + r(t)h(t) \frac{\pi}{\pi} \frac{r(t)}{r(t)} \right) = 2\pi \left(r^2(t) + \frac{V(t)}{\pi r(t)} \right)$$

$$A'(t) = 2\pi \left[2r(t)r'(t) + \frac{V'(t)\pi r(t) - V(t)\pi r'(t)}{\pi^2 r^2(t)} \right] \quad \forall t$$

Let $t=t_0$ denotes the specified moment. Then

$$A'(t_0) = 2\pi \left[2 \cdot 5 \cdot \frac{1}{2} + \frac{100\pi \cdot \pi \cdot 5 - \overbrace{V(t_0)}^{r^2(t_0)\pi h(t_0) = 5^2 \cdot \pi \cdot 10 = 250\pi} \pi \frac{1}{2}}{\pi^2 \cdot 25} \right]$$

$$= 2\pi \left[5 + \frac{500 - 125}{25} \right] = 2\pi \left[5 + \frac{375}{25} \right] = 2\pi(5+15)$$

$$= 40\pi \text{ cm}^2/\text{sec}.$$

(14 pts) 4. Let $f(x) = \begin{cases} e^x(x^2 + a) & \text{if } x < 0, \\ 1 & \text{if } x = 0, \\ bx^2 + 1 & \text{if } x > 0. \end{cases}$

If possible, determine a, b so that $f'(0)$ exists. If not possible explain why?

$(f'(x) \text{ exists at } x=x_0) \Rightarrow (f(x) \text{ is continuous at } x=x_0)$

check the continuity at $x_0 = 0$ ($f(x)$ is continuous $\forall x_0 \neq 0, \forall a, b \in \mathbb{R}$)

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} bx^2 + 1 = 1$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} e^x(x^2 + a) = 0$$

$$a = 1 \quad (\Rightarrow \lim_{x \rightarrow 0} f(x) = f(0) = 1 \checkmark)$$

Now let find $f'(0)$:

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{bh^2 + 1 - 1}{h} = \lim_{h \rightarrow 0^+} bh = 0 \quad \forall b$$

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{e^h(h^2 + 1) - 1}{h} = \lim_{h \rightarrow 0^-} \frac{g(0+h) - g(0)}{h} = g'(0)$$

$= 1 \quad \forall b$

where $g(x) = e^x(x^2 + 1)$

$(\Rightarrow g'(x) = e^x(x^2 + 1) + e^x \cdot 2x$
 $\Rightarrow g'(0) = 1)$

$a = 1$ &

whatever b is chosen, $f'(0)$ does NOT exist.

$\therefore$ There is no such a, b .

(10 pts) 5. If $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies $f'(x) = e^{-x^2}$ for all x and $f(0) = 0$ then prove that

$$|f(x)| \leq |x| \text{ for all } x.$$

$f'(x) = e^{-x^2} \forall x \Rightarrow f(x)$ is continuous and differentiable on $\mathbb{R}$

For any $a \in \mathbb{R} \setminus \{0\}$,

if $a > 0$ then $f(x)$ is continuous on $[0, a]$ and differentiable on $(0, a)$

if $a < 0$ " " " " $[a, 0]$ " " " $(a, 0)$

Then by Mean Value Theorem

$$\frac{f(a) - f(0)}{a - 0} = f'(c) \text{ for some } c \text{ between } 0 \text{ \& } a$$

(no matter if $a > 0$ or $a < 0$)

$$\Rightarrow \frac{f(a)}{a} = e^{-c^2} = \frac{1}{e^{c^2}} \quad \left. \begin{matrix} c^2 > 0 \\ (c \neq 0) \\ e^0 = 1 \end{matrix} \right\} \Rightarrow 0 < \frac{1}{e^{c^2}} < 1$$

$$\Rightarrow f(a) = a \cdot e^{-c^2} \Rightarrow |f(a)| = |a| \frac{1}{e^{c^2}} < |a|$$

$$\Rightarrow |f(a)| < |a| \text{ when } a \neq 0$$

$(a = 0 \Rightarrow |f(0)| = |0| = 0)$

$$\therefore |f(x)| \leq |x| \quad \forall x \in \mathbb{R}$$

(7+7+7+7 pts) 6.

(a) Find y' if $y = \frac{1}{\ln(\arcsin x)} = [\ln(\arcsin x)]^{-1}$

$$y'(x) = -1 \cdot \frac{1}{[\ln(\arcsin x)]^2} \cdot \frac{1}{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}}$$

(b) Find y' if $y = (x^2 + 119)^{\sqrt{x}} \Rightarrow \ln y = \sqrt{x} \ln(x^2 + 119)$

$$\Rightarrow \frac{y'}{y} = \frac{1}{2\sqrt{x}} \ln(x^2 + 119) + \sqrt{x} \frac{2x}{x^2 + 119}$$

$$\Rightarrow y' = (x^2 + 119)^{\sqrt{x}} \left[\frac{1}{2\sqrt{x}} \ln(x^2 + 119) + \frac{2x\sqrt{x}}{x^2 + 119} \right]$$

(c) Find $\frac{d}{dx} f^{-1}(x)$ at $x = 2$ where $f(x) = x^3 + 5x + 2$.

$$\begin{aligned} (f^{-1})'(2) &= \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(0)} \\ &= \frac{1}{(3x^2 + 5)|_{x=0}} = \frac{1}{5} \end{aligned}$$

$$\begin{aligned} (f^{-1} \circ f)'(x) &= (f^{-1})'(f(x)) \cdot f'(x) \\ (\text{Id}(x))' &= x' = 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow (f^{-1})'(f(x)) &= \frac{1}{f'(x)} \\ \text{or equivalently} \\ (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

(d) Let f and g be everywhere differentiable functions satisfying $f(1) = 1$, $f'(1) = -2$, $f''(1) = 4$ and $g(1) = 0$, $g'(1) = 3$, $g''(1) = -2$. Compute y'' at $x = 1$ where $y = f(f(x) + 2g(x))$.

$$y' = f'(f(x) + 2g(x)) \cdot (f'(x) + 2g'(x))$$

$$y'' = f''(f(x) + 2g(x)) \cdot (f'(x) + 2g'(x))^2 + f'(f(x) + 2g(x)) (f''(x) + 2g''(x))$$

at $x = 1$ we have

$$\begin{aligned} y'' &= f''(1 + 2 \cdot 0) \cdot (-2 + 2 \cdot 3)^2 + f'(1 + 2 \cdot 0) (4 + 2(-2)) \\ &= 4 \cdot 16 = 64 \end{aligned}$$