MATH 117 & MATH 119

RECITATION 6

Chapter 5 : Integration

5.5 The Fundamental Theorem of Calculus
5.6 The Method of Substitution
5.7 Areas of Plane Regions

5.5 The Fundamental Theorem of Calculus

Theorem: The Fundamental Theorem of Calculus

Suppose that f is a continuous function on I and $a \in I$.

(i) Let the function F be defined on I by $F(x) = \int_a^x f(t)\,dt$. Then F is differentiable on I,

and $F'(x) = f(x)$. Thus, F is an anti-derivative of f on I.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

(ii) If $G(x)$ is any anti-derivative of $f(x)$ on I, $G'(x) = f(x)$ then for any $b \in I$

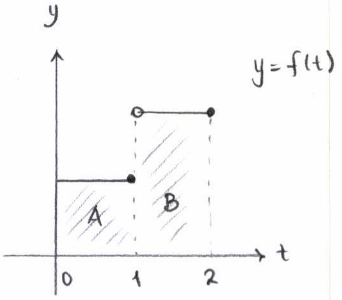
$$\int_a^b f(x)\,dx = G(b) - G(a).$$

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)$$

1. Let $f(t) = \begin{cases} 1 & \text{if } 0 \leqslant t \leqslant 1 \\ 2 & \text{if } 1 < t \leqslant 2 \end{cases}$ and $F(x) = \int_0^x f(t)\,dt$. If exists, find $F'(1)$.

Solution:



$F(x)$ can be defined also as $F(x) = \begin{cases} \text{area}(A) = 1, & 0 \leqslant x \leqslant 1 \\ \text{area}(B) = 2, & 1 < x \leqslant 2 \end{cases}$

to find $F'(1)$

$$F'_+(1) = \lim_{x \to 1^+} \frac{F(x) - F(1)}{x - 1} = \lim_{x \to 1^+} \frac{2 - 1}{x - 1} = \infty$$

$$\neq \qquad F'(1) \text{ does not exist.}$$

$$F'_-(1) = \lim_{x \to 1^-} \frac{F(x) - F(1)}{x - 1} = \lim_{x \to 1^-} \frac{1 - 1}{x - 1} = 0$$

! The Fundamental Theorem of calculus cannot be applied since f is not continuous.

2. Evaluate the following limits and derivatives.

(a) $\dfrac{d^2}{dx^2}\displaystyle\int_0^x e^{\sin(t)}\,dt$

(b) $\displaystyle\lim_{x\to 0}\frac{1}{x}\cdot\int_{\sin x}^{x^3}\sqrt{t^2+1}\,dt$

(c) $\displaystyle\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\cdots+\frac{n}{2n^2}\right)$

(d) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{4n}\right)$

Solution:

(a) $\displaystyle\frac{d}{dx}\left[\frac{d}{dx}\int_0^x e^{\sin(t)}\,dt\right]=\frac{d}{dx}\left[e^{\sin x}\right]$ by the FTC ($f(t)=e^{\sin(t)}$ is continuous)

$= e^{\sin x}\cdot\cos x$

(b) $\displaystyle\lim_{x\to 0}\frac{\int_{\sin x}^{x^3}\sqrt{t^2+1}\,dt}{x}$ $\left[\frac{0}{0}\text{ - type}\right]$

$\overset{L'H}{=}\displaystyle\lim_{x\to 0}\frac{\frac{d}{dx}\int_{\sin x}^{x^3}\sqrt{t^2+1}\,dt}{\frac{d}{dx}x}=\lim_{x\to 0}\frac{\sqrt{x^6+1}\cdot(3x^2)-\sqrt{\sin^2x+1}\cdot\cos x}{1}$

$= -1$

(c) $\displaystyle\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\cdots+\frac{n}{2n^2}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i^2}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\left(\frac{i}{n}\right)^2}\cdot\frac{1}{n}$

Choose $f(x)=\dfrac{1}{1+x^2}$, $\Delta x=\dfrac{1}{n}$, $[0,1]$ $\displaystyle=\int_0^1\frac{1}{1+x^2}\,dx$

(integeable)

Let $g(x)=\arctan x$ then $g'(x)=\dfrac{1}{1+x^2}$ so $g(x)$ is anti-derivative of $\dfrac{1}{1+x^2}$

By the FTC, $\displaystyle\int_0^1\frac{1}{1+x^2}\,dx=g(1)-g(0)=\arctan 1-\arctan 0$

$=\dfrac{\pi}{4}$

(d) $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{4n}\right)=\lim_{n\to\infty}\sum_{i=1}^{3n}\frac{1}{n+i}=\lim_{n\to\infty}\sum_{i=1}^{3n}\frac{1}{1+\frac{i}{n}}\cdot\frac{1}{n}$

Let $m=3n$, as $n\to\infty$ $m\to\infty$

$$\lim_{m\to\infty}\sum_{i=1}^{m}\frac{1}{\frac{1}{3}+\frac{i}{m}}\cdot\frac{1}{m} = \lim_{m\to\infty}\sum f(x_i^*).\Delta x_i = \int_0^1 \frac{1}{\frac{1}{3}+x}\,dx$$

Choose $f(x)=\frac{1}{\frac{1}{3}+x}$, which is integrable, $\Delta x=\frac{1}{m}$, $[0,1]$

$g(x)=\ln\left(\frac{1}{3}+x\right)$ then $g'(x)=\frac{1}{\frac{1}{3}+x}$ so $g(x)$ is an anti-derivative of $\frac{1}{\frac{1}{3}+x}$ so by the FTC

$$\int_0^1\frac{1}{\frac{1}{3}+x}\,dx = g(1)-g(0) = \ln\left(\frac{1}{3}+1\right)-\ln\left(\frac{1}{3}+0\right)=\ln 4$$

3. Evaluate $\displaystyle\lim_{x\to 0}\frac{\int_{-2}^{0}\sin(t^2)\,dt-\int_{-2}^{x^2}\sin(t^2)\,dt}{\int_{\cos x}^{1}e^{\sqrt{t}-1}\,dt}$

Solution:

$\left[\frac{0}{0}\text{-type}\right]$ Apply L'Hôpital Rule

$$\lim_{x\to 0}\frac{\int_{-2}^{0}\sin(t^2)\,dt-\int_{-2}^{x^2}\sin(t^2)\,dt}{\int_{\cos x}^{1}e^{\sqrt{t}-1}\,dt}\overset{L'H}{=}\lim_{x\to 0}\frac{\frac{d}{dx}\Big(\underbrace{\int_{-2}^{0}\sin(t^2)\,dt}_{\text{constant}}\Big)-\frac{d}{dx}\left(\int_{-2}^{x^2}\sin(t^2)\,dt\right)}{\frac{d}{dx}\left(\int_{\cos x}^{1}e^{\sqrt{t}-1}\,dt\right)}$$

by the FTC

$$=\lim_{x\to 0}\frac{0-\sin(x^4).2x}{-e^{\sqrt{\cos x}-1}(-\sin x)}=\lim_{x\to 0}\left[-\underbrace{\frac{2\sin(x^4)}{e^{\sqrt{\cos x}-1}}}_{0}\cdot\underbrace{\frac{1}{\frac{\sin x}{x}}}_{1}\right]$$

$$=0$$