MATH 117 & MATH 119

RECITATION 6

Chapter 5 : Integration

5.5 The Fundamental Theorem of Calculus
5.6 The Method of Substitution
5.7 Areas of Plane Regions

5.5 The Fundamental Theorem of Calculus

Theorem: The Fundamental Theorem of Calculus

Suppose that f is a continuous function on I and $a \in I$.

(i) Let the function F be defined on I by $F(x) = \int_a^x f(t)\,dt$. Then F is differentiable on I, and $F'(x) = f(x)$. Thus, F is an anti-derivative of f on I.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

(ii) If $G(x)$ is any anti-derivative of $f(x)$ on I, $G'(x) = f(x)$ then for any $b \in I$

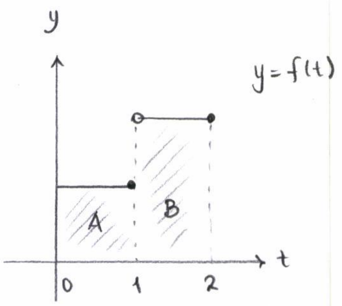
$$\int_a^b f(x)\,dx = G(b) - G(a).$$

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)$$

1. Let $f(t) = \begin{cases} 1 & \text{if } 0 \leqslant t \leqslant 1 \\ 2 & \text{if } 1 < t \leqslant 2 \end{cases}$ and $F(x) = \int_0^x f(t)\,dt$. If exists, find $F'(1)$.

Solution:



$F(x)$ can be defined also as $F(x) = \begin{cases} \text{area}(A) = 1, & 0 \leqslant x \leqslant 1 \\ \text{area}(B) = 2, & 1 < x \leqslant 2 \end{cases}$

to find $F'(1)$

$$F'_{+}(1) = \lim_{x \to 1^+} \frac{F(x) - F(1)}{x - 1} = \lim_{x \to 1^+} \frac{2-1}{x-1} = \infty$$

$$\neq \qquad F'(1) \text{ does not exist.}$$

$$F'_{-}(1) = \lim_{x \to 1^-} \frac{F(x) - F(1)}{x - 1} = \lim_{x \to 1^-} \frac{1-1}{x-1} = 0$$

! The Fundamental Theorem of calculus cannot be applied since f is not continuous.

2. Evaluate the following limits and derivatives.

(a) $\dfrac{d^2}{dx^2}\displaystyle\int_0^x e^{\sin(t)}\,dt$

(b) $\displaystyle\lim_{x\to 0} \frac{1}{x}\cdot\int_{\sin x}^{x^3}\sqrt{t^2+1}\,dt$

(c) $\displaystyle\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\cdots+\frac{n}{2n^2}\right)$

(d) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{4n}\right)$

Solution:

(a) $\dfrac{d}{dx}\left[\dfrac{d}{dx}\displaystyle\int_0^x e^{\sin(t)}\,dt\right] = \dfrac{d}{dx}\left[e^{\sin x}\right]$ by the FTC ($f(t)=e^{\sin(t)}$ is continuous)

$= e^{\sin x}\cdot\cos x$

(b) $\displaystyle\lim_{x\to 0}\frac{\int_{\sin x}^{x^3}\sqrt{t^2+1}\,dt}{x}$ $\left[\frac{0}{0}\text{-type}\right]$

$\overset{L'H}{=} \displaystyle\lim_{x\to 0}\frac{\frac{d}{dx}\int_{\sin x}^{x^3}\sqrt{t^2+1}\,dt}{\frac{d}{dx}x} = \lim_{x\to 0}\frac{\sqrt{x^6+1}\,.(3x^2)-\sqrt{\sin^2 x+1}\,.\cos x}{1}$

$= -1$

(c) $\displaystyle\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\cdots+\frac{n}{2n^2}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\left(\frac{i}{n}\right)^2}\cdot\frac{1}{n}$

Choose $f(x)=\dfrac{1}{1+x^2}$, $\Delta x=\dfrac{1}{n}$, $[0,1]$ $\qquad = \displaystyle\int_0^1\frac{1}{1+x^2}\,dx$

(integeable)

Let $g(x)=\arctan x$ then $g'(x)=\dfrac{1}{1+x^2}$ so $g(x)$ is anti-derivative of $\dfrac{1}{1+x^2}$

By the FTC, $\displaystyle\int_0^1\frac{1}{1+x^2}\,dx = g(1)-g(0) = \arctan 1 - \arctan 0$

$= \dfrac{\pi}{4}$

(d) $\lim_{n \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{4n} \right) = \lim_{n \to \infty} \sum_{i=1}^{3n} \frac{1}{n+i} = \lim_{n \to \infty} \sum_{i=1}^{3n} \frac{1}{1+\frac{i}{n}} \cdot \frac{1}{n}$

Let $m = 3n$, as $n \to \infty$ $m \to \infty$

$$\lim_{m \to \infty} \sum_{i=1}^{m} \frac{1}{\frac{1}{3} + \frac{i}{m}} \cdot \frac{1}{m} = \lim_{m \to \infty} \sum f(x_i^*) . \Delta x_i = \int_0^1 \frac{1}{\frac{1}{3}+x} dx$$

Choose $f(x) = \frac{1}{\frac{1}{3}+x}$, which is integrable, $\Delta x = \frac{1}{m}$, $[0,1]$

$g(x) = \ln\left(\frac{1}{3}+x\right)$ then $g'(x) = \frac{1}{\frac{1}{3}+x}$ so $g(x)$ is an anti-derivative of $\frac{1}{\frac{1}{3}+x}$ so by the FTC

$$\int_0^1 \frac{1}{\frac{1}{3}+x} dx = g(1) - g(0) = \ln\left(\frac{1}{3}+1\right) - \ln\left(\frac{1}{3}+0\right) = \ln 4$$

3. Evaluate $\lim_{x \to 0} \frac{\int_{-2}^{0} \sin(t^2) dt - \int_{-2}^{x^2} \sin(t^2) dt}{\int_{\cos x}^{1} e^{\sqrt{t}-1} dt}$

Solution:

$\left[\frac{0}{0} - \text{type}\right]$ Apply L'Hôpital Rule

$$\lim_{x \to 0} \frac{\int_{-2}^{0} \sin(t^2) dt - \int_{-2}^{x^2} \sin(t^2) dt}{\int_{\cos x}^{1} e^{\sqrt{t}-1} dt} \overset{L'H}{=} \lim_{x \to 0} \frac{\frac{d}{dx}\Big(\overbrace{\int_{-2}^{0} \sin(t^2) dt}^{\text{constant}}\Big) - \frac{d}{dx}\left(\int_{-2}^{x^2} \sin(t^2) dt\right)}{\frac{d}{dx}\left(\int_{\cos x}^{1} e^{\sqrt{t}-1} dt\right)}$$

by the FTC

$$= \lim_{x \to 0} \frac{0 - \sin(x^4) . 2x}{-e^{\sqrt{\cos x}-1} (-\sin x)} = \lim_{x \to 0} \left[- \underbrace{\frac{2 \sin(x^4)}{e^{\sqrt{\cos x}-1}}}_{0} \cdot \underbrace{\frac{1}{\frac{\sin x}{x}}}_{1} \right]$$

$$= 0$$

5.6 The Method of Substitution

Theorem: Suppose that g is a differentiable function on $[a,b]$ that satisfies $g(a)=A$ and $g(b)=B$. Also suppose that f is continuous on the range of g. Then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_A^B f(u)\,du$$

4. Evaluate the following definite or indefinite integrals.

(a) $\displaystyle\int \frac{2x+3}{4x^2+4x+3}\,dx$

Solution:

$$\int \frac{2x+3}{4x^2+4x+3}\,dx = \frac{1}{4}\int \frac{4(2x+1)}{4x^2+4x+3}\,dx + \int \frac{3}{4x^2+4x+3}\,dx$$

$$u = 4x^2+4x+3$$
$$du = 8x+4$$

$$= \frac{1}{4}\int \frac{1}{u}\,du + \frac{3}{4}\int \frac{1}{x^2+x+\frac{1}{4}+\frac{1}{2}} = \frac{1}{4}\int \frac{1}{u}\,du + \frac{3}{4}\int \frac{1}{\frac{1}{2}+(x+\frac{1}{2})^2}\,dx$$

$$= \frac{1}{4}\int \frac{1}{u}\,du + \frac{3}{4}\,2\int \frac{1}{1+\left(\frac{x+1/2}{\sqrt{2}}\right)^2}\,dx = \frac{1}{4}\int \frac{1}{u}\,du + \frac{3\sqrt{2}}{4}\int \frac{1}{1+\left(\frac{x+1/2}{\sqrt{2}}\right)^2}\,\sqrt{2}\,dx$$

$$t = \frac{x}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \Rightarrow dt = \frac{1}{\sqrt{2}}$$

$$= \frac{1}{4}\int \frac{1}{u}\,du + \frac{3\sqrt{2}}{4}\int \frac{1}{1+t^2}\,dt = \frac{1}{4}\ln|u| + \frac{3\sqrt{2}}{4}\arctan t + C$$

$$= \frac{1}{4}\ln|4x^2+4x+3| + \frac{3\sqrt{2}}{4}\arctan\left(\frac{x+1/2}{\sqrt{2}}\right) + C$$

(b) $\displaystyle\int \frac{1}{e^x+e^{-x}}\,dx$

Solution:

$$\int \frac{1}{e^x+\frac{1}{e^x}}\,dx = \int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{1}{1+u^2}\,du = \arctan u + C = \arctan(e^x) + C$$

$$u = e^x$$
$$du = e^x\,dx$$

(c) $\int \frac{t}{\sqrt{9-t^4}}\,dt$

Solution:

$u = t^2 \Rightarrow du = 2t\,dt$

$$\frac{1}{2}\int \frac{2t}{\sqrt{9-t^4}}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{9-u^2}}\,du = \frac{1}{2}\int \frac{1}{3}\cdot\frac{1}{\sqrt{1-\left(\frac{u}{3}\right)^2}}\,du \qquad x = \frac{u}{3} \Rightarrow dx = \frac{1}{3}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\arcsin x + C = \frac{1}{2}\arcsin\frac{u}{3} + C$$

$$= \frac{1}{2}\arcsin\left(\frac{t^2}{3}\right) + C$$

(d) $\int \cos^4 x\,dx$

Solution:

$$\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{\cos 2x + 1}{2} \;;\; \cos 4x = 2\cos^2(2x) - 1 \Rightarrow \cos^2(2x) = \frac{\cos 4x + 1}{2}$$

$$\int \cos^4 x\,dx = \int \left(\frac{\cos 2x + 1}{2}\right)^2 dx = \frac{1}{4}\int \left(\cos^2(2x) + 2\cos(2x) + 1\right) dx$$

$$= \frac{1}{4}\int \left[\frac{\cos 4x}{2} + \frac{1}{2} + 2\cos 2x + 1\right] dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x + \frac{\sin 4x}{8} + \sin 2x\right] + C$$

(e) $\int \sin^3 x . \sec^5 x\,dx$

Solution:

$$\int \frac{\sin^3 x}{\cos^5 x}\,dx = \int \frac{\sin^2 x}{\cos^5 x}\,.\sin x\,dx = \int \frac{1-\cos^2 x}{\cos^5 x}\,.\sin x\,dx \qquad \begin{aligned} u &= \cos x \\ du &= -\sin x\,dx \end{aligned}$$

$$= \int \left(\cos^{-5} x - \cos^{-3} x\right) \sin x\,dx = \int \left(-u^{-5} + u^{-3}\right) du$$

$$= -\frac{u^{-4}}{(-4)} + \frac{u^{-2}}{(-2)} + C = \frac{1}{4u^4} - \frac{1}{2u^2} + C$$

$$= \frac{1}{4\cos^4 x} - \frac{1}{2\cos^2 x} + C$$

(f) $\int \sec^3 x . \tan^5 x \, dx$

Solution:

$\frac{d}{dx} \tan x = \sec^2 x = 1 + \tan^2 x \quad , \quad \frac{d}{dx} \sec x = \sec x . \tan x$

$$\int \sec^3 x . \tan^5 x dx = \int \sec^2 x . \tan^4 x . \sec x . \tan x \, dx$$

$u = \sec x \Rightarrow du = \tan x \sec x dx$

$\tan^2 x = \sec^2 x - 1$

$$= \int u^2 (u^2-1)^2 du = \int u^2 (u^4 - 2u^2 + 1) du$$

$$= \int (u^6 - 2u^4 + u^2) du = \frac{u^7}{7} - 2\frac{u^5}{5} + \frac{u^3}{3} + C$$

$$= \frac{\sec^7 x}{7} - \frac{2}{5} \sec^5 x + \frac{\sec^3 x}{3} + C$$

(g) $\int \cos 3x . \sin 5x \, dx$

Solution:

$\sin a . \cos b = \frac{1}{2} [\sin(a+b) + \sin(a-b)]$

$$\int \cos 3x . \sin 5x \, dx = \frac{1}{2} \int [\sin 8x + \sin 2x] dx = \frac{1}{2} \left[-\frac{\cos 8x}{8} - \frac{\cos 2x}{2} \right] + C$$

(h) $\int_1^e \frac{\sin^3(\ln x) . \cos^3(\ln x) dx}{x}$

Solution:

$u = \sin(\ln x) \Rightarrow du = \cos(\ln x) . \frac{1}{x} dx$

$u(1) = \sin(\ln 1) = \sin 0 = 0$

$u(e) = \sin(\ln e) = \sin 1$

$$\int_1^e \sin^3(\ln x) . \cos^2(\ln x) \frac{\cos(\ln x)}{x} dx = \int_0^{\sin 1} u^3 (1 - u^2) du$$

$$= \int_0^{\sin 1} (u^3 - u^5) du = \left(\frac{u^4}{4} - \frac{u^6}{6} \right) \Big|_0^{\sin 1}$$

$$= \left[\frac{(\sin 1)^4}{4} - \frac{(\sin 1)^6}{6} \right] - [0 - 0] = \frac{(\sin 1)^4}{4} - \frac{(\sin 1)^6}{6}$$

(i) $\int \frac{x^2}{\sqrt{2x-1}}\,dx$

Solution:

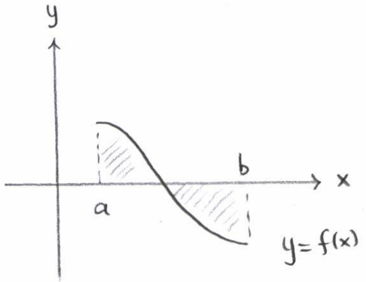
$2x-1 = u \Rightarrow \quad 2\,dx = du$

$x = \frac{u+1}{2} \Rightarrow x^2 = \frac{1}{4}(u^2+2u+1)$

$$\frac{1}{2}\int \frac{x^2}{\sqrt{2x-1}}\,2\,dx = \frac{1}{2}\int \frac{1}{4}(u^2+2u+1)\cdot\frac{1}{\sqrt{u}}\,du = \frac{1}{8}\int \left[u^{3/2}+2u^{1/2}+u^{-1/2}\right]du$$

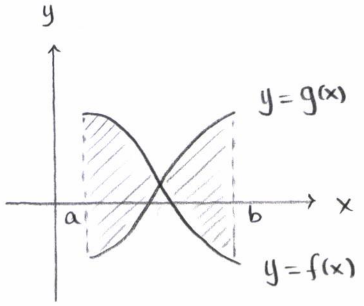
$$= \frac{1}{8}\left[\frac{u^{5/2}}{\frac{5}{2}} + 2\frac{u^{3/2}}{\frac{3}{2}} + \frac{u^{1/2}}{\frac{1}{2}}\right] + C = \frac{1}{8}\left[\frac{2}{5}(2x-1)^{5/2} + \frac{4}{3}(2x-1)^{3/2} + 2(2x-1)^{1/2}\right] + C$$

5.7 Areas of Plane Regions



Area of the region bounded by the graph of $y=f(x)$ and x-axis over $[a,b]$:

$$A = \int_a^b |f(x)|\,dx$$



The area lying between the graphs $y=f(x)$ and $y=g(x)$ over $[a,b]$:

$$A = \int_a^b |f(x) - g(x)|\,dx$$

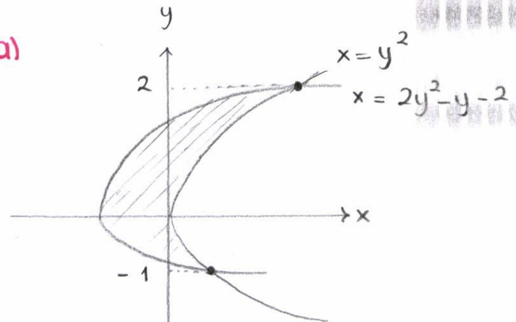
5. Sketch and find the areas of the plane regions bounded by the curves

(a) $x = y^2$ and $x = 2y^2 - y - 2$

(b) $y = (x^2-1)^2$ and $y = 1 - x^2$

Solution :

(a)



Find intersection points of these curves:

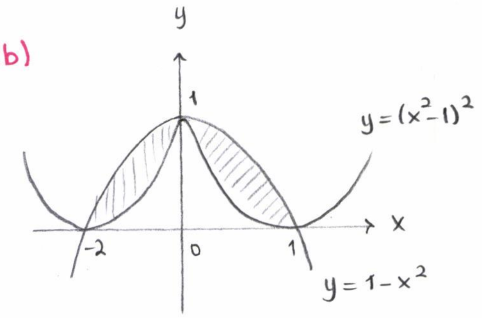
$y^2=2y-y-2 \Rightarrow y^2-y-2=0 \Rightarrow (y-2)(y+1)=0 \Rightarrow y=2 \quad y=-1$

Area is

$$A = \int_{-1}^{2} \left| y^2-(2y^2-y-2) \right| dy = \int_{-1}^{2} \left[-y^2+y+2\right] dy$$

$$= \left[-\frac{y^3}{3}+\frac{y^2}{2}+2y\right]\Big|_{-1}^{2} = \left(-\frac{8}{3}+2+4\right)-\left(\frac{1}{3}+\frac{1}{2}-2\right) = \frac{9}{2}$$

(b)



The intersections of the curves:

$(x^2-1)^2 = 1-x^2 \Rightarrow x=0 \text{ , } x=1 \text{ , } x=-1$

The Region is symmetric about y-axis so we can take the integeal over [0,1] to find half of the Region:

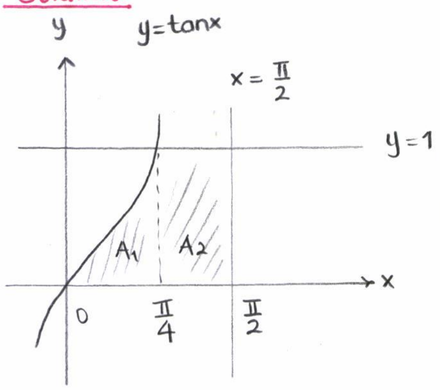
$$A = \int_{0}^{1} \left| (1-x^2)-(x^2-1)^2 \right| dx = \int_{0}^{1} \left[(1-x^2)-(x^4-2x^2+1)\right] dx$$

$$= \int_{0}^{1} [-x^4+x^2]\, dx = \left[-\frac{x^5}{5}+\frac{x^3}{3}\right]\Big|_{0}^{1} = \frac{2}{15}$$

The total area $= 2 \cdot \frac{2}{15} = \frac{4}{15}$

6. Find the area of the plane Region bounded by the curves $y=\tan x$, $y=0$, $y=1$ and $x=\frac{\pi}{2}$.

Solution:



The intersection of $y=\tan x$ and $y=1$ is $x=\frac{\pi}{4}$

$$\text{The total area} = A_1+A_2 = \int_{0}^{\frac{\pi}{4}} \tan x \, dx + \frac{\pi}{4}$$

$$= -\ln|\cos x| \Big|_{0}^{\frac{\pi}{4}} + \frac{\pi}{4}$$

$$= -\ln\left|\cos\frac{\pi}{4}\right| + \ln|\cos 0| + \frac{\pi}{4}$$

$$= \frac{\pi}{4} - \ln\frac{\sqrt{2}}{2}$$