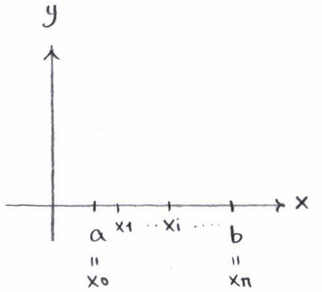
MATH117 & MATH 119

RECITATION 5

Chapter 5 : Integration

5.1 Sums and Sigma Notation
5.2 Areas as Limits of Sums
5.3 The Definite Integral
5.4 Properties of Definite Integral

Partition: Let P be a finite set of points $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ where $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$

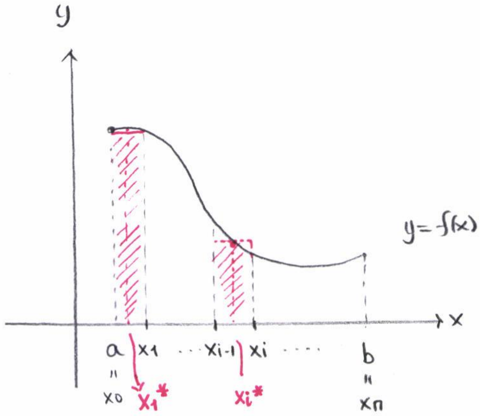


Such a set P is called a partition of $[a,b]$, divides $[a,b]$ into n subintervals of which the ith is $[x_{i-1}, x_i]$.

The length of ith subinterval is $\Delta x_i = x_{i-1} - x_i$, $i \in \{1,2,\dots n\}$

The norm of the partition P is $\|P\| = \max\limits_{1 \le i \le n} \Delta x_i$

Riemann Sum: Let $f(x)$ be a continuous function defined on $[a,b]$ and $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$.



For arbitrary points on each subintervals, $x_i^* \in [x_{i-1}, x_i]$, the Riemann Sum of f corresponding the partition P

$$R(f, P, x_i^*) = \sum_{i=1}^{n} f(x_i^*) . \Delta x_i$$

$$= f(x_1^*) \Delta x_1 + f(x_2^*) \Delta x_2 + \dots + f(x_n^*) \Delta x_n .$$

Since f is continuous on $[a,b]$, on each subinterval $[x_{i-1}, x_i]$ it has absolute maximum and minimum values. Thus, there are numbers l_i and u_i in $[x_{i-1}, x_i]$ such that $f(l_i) \le f(x) \le f(u_i)$ whenever $x_{i-1} \le x \le x_i$.

The Lower Riemann Sum of f : $L(f,P) = f(l_1)\Delta x_1 + f(l_2).\Delta x_2 + \dots + f(l_n)\Delta x_n$

$$= \sum_{i=1}^{n} f(l_i) \Delta x_i$$

The Upper Riemann Sum of f : $U(f,P) = f(u_1).\Delta x_1 + f(u_2).\Delta x_2 + \dots + f(u_n).\Delta x_n$

$$= \sum_{i=1}^{n} f(u_i). \Delta x_i$$

The Definite Integral : Suppose there is exactly one number I such that for every partition P of $[a,b]$ we have $L(f,P) \leqslant I \leqslant U(f,P)$, then we say that the function f is integrable on $[a,b]$, I is the definite integral of f on $[a,b]$.

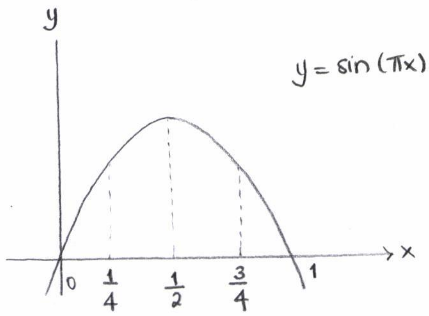
$$I = \int_a^b f(x)\,dx$$

If f is integrable on $[a,b]$, then $\lim\limits_{\substack{n \to \infty \\ \|P\| \to 0}} R(f,P,c) = \int_a^b f(x)\,dx$.

1. Compute the lower and upper Riemann sum of the function $f(x) = \sin(\pi . x)$ over the interval $[0,1]$ using the partition of $[0,1]$ into four subintervals of equal length.

Solution:

Sketch the graph of the function $f(x) = \sin(\pi x)$ over $[0,1]$.



The partition $P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ divides $[0,1]$ into four subintervals of equal length.

$\Delta x_i = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$ for each subinterval.

To compute lower Riemann sum, we need to find absolute minimum value of f on each subinterval. On $[0,\frac{1}{4}]$, $f(0)$; on $[\frac{1}{4},\frac{1}{2}]$, $f(\frac{1}{4})$; on $[\frac{1}{2},\frac{3}{4}]$, $f(\frac{3}{4})$ and on $[\frac{3}{4},1]$, $f(1)$ are absolute minimum values of f. Thus the lower Riemann sum of f corresponding P is

$$L(f,P) = \sum_{i=1}^{4} f(l_i)\,\Delta x_i = f(0)\,\Delta x_1 + f\left(\tfrac{1}{4}\right)\Delta x_2 + f\left(\tfrac{3}{4}\right)\Delta x_3 + f(1)\,\Delta x_4$$

$$= \frac{1}{4}\left(f(0) + f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right) + f(1)\right) = \frac{1}{4}\left(0 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 0\right) = \frac{\sqrt{2}}{4}$$

To compute upper Riemann sum, we need to find absolute maximum value of f on each subinterval. On $[0,\frac{1}{4}]$, $f(\frac{1}{4})$; on $[\frac{1}{4},\frac{1}{2}]$, $f(\frac{1}{2})$; on $[\frac{1}{2},\frac{3}{4}]$, $f(\frac{1}{2})$ and on $[\frac{3}{4},1]$, $f(\frac{3}{4})$ are absolute maximum values of f. Thus the upper Riemann sum of f corresponding P is

$$U(f,P) = \sum_{i=1}^{4} f(u_i)\,\Delta x_i = f\left(\tfrac{1}{4}\right)\Delta x_1 + f\left(\tfrac{1}{2}\right)\Delta x_2 + f\left(\tfrac{1}{2}\right)\Delta x_3 + f\left(\tfrac{3}{4}\right)\Delta x_4$$

$$= \frac{1}{4}\left(f\left(\tfrac{1}{4}\right) + f\left(\tfrac{1}{2}\right) + f\left(\tfrac{1}{2}\right) + f\left(\tfrac{3}{4}\right)\right) = \frac{1}{4}\left(\frac{\sqrt{2}}{2} + 1 + 1 + \frac{\sqrt{2}}{2}\right) = \frac{1}{2} + \frac{\sqrt{2}}{4}$$

2. Show that for any continuous increasing function $f(x)$ defined on $\mathbb{R}$ such that $f(0)=0$, $f(1/4)=2$, $f(1/2)=4$, $f(3/4)=50$ and $f(1)=120$, we have that $14 \leq \int_0^1 f(x)\,dx \leq 44$.

Solution:

Since f is a continuous function, it is integrable on $[0,1]$.

Moreover, for every partition of $[0,1]$ we have the following inequality since f is integrable

$$L(f,P) \leq \int_0^1 f(x)\,dx \leq U(f,P)$$

Let $P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ be a partition for $[0,1]$ and each subinterval has length

$$\Delta x_i = \frac{1-0}{4} = \frac{1}{4}.$$

To write lower Riemann sum, since f is increasing we need to choose left end points of subintervals:

$$L(f,P) = \sum_{i=1}^{4} f(l_i)\,\Delta x_i = f(0)\,\Delta x_1 + f\left(\tfrac{1}{4}\right)\Delta x_2 + f\left(\tfrac{1}{2}\right)\Delta x_3 + f(1)\,\Delta x_4$$

$$= \frac{1}{4}\left(0+2+4+50\right) = 14$$

To write upper Riemann sum, since f is increasing we need to choose right end points of subintervals:

$$U(f,P) = \sum_{i=1}^{4} f(u_i)\,\Delta x_i = f\left(\tfrac{1}{4}\right)\Delta x_1 + f\left(\tfrac{1}{2}\right)\Delta x_2 + f\left(\tfrac{3}{4}\right)\Delta x_3 + f(1)\,\Delta x_4$$

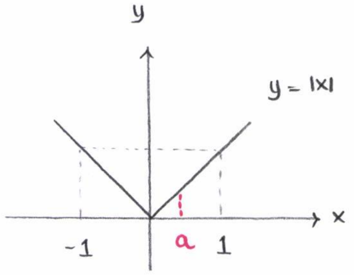
$$= \frac{1}{4}\left(2+4+50+120\right) = 44$$

Therefore;

$$L(f,P) \leq \int_0^1 f(x)\,dx \leq U(f,P) \Rightarrow 14 \leq \int_0^1 f(x)\,dx \leq 44.$$

3. Find a partition of $[-1,1]$ into two subintervals such that the lower Riemann sum for the function $f(x) = |x|$ over $[-1,1]$ corresponding to the partition is $\frac{2}{9}$.

Solution:



Consider the partition $P = \{-1, a, 1\}$ and assume that $a > 0$.

On $[-1,a]$, $f(0)$ is absolute minimum value of f and on $[a,1]$, $f(a)$ is absolute minimum value of f.

The lower Riemann sum of f corresponding to P is

$$L(f,P) = \sum_{i=1}^{2} f(l_i)\,\Delta x_i = f(0)\,\Delta x_1 + f(a)\,.\,\Delta x_2$$

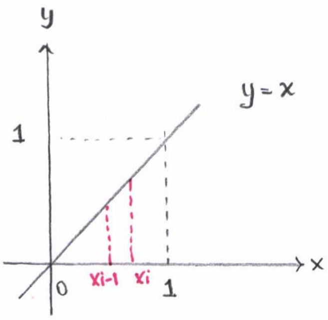
$$= 0\,.\,(a-(-1)) + |a|\,.\,(1-a) \qquad (\text{since } a>0,\ |a| = a)$$

$$= a - a^2$$

$$a - a^2 = \frac{2}{9} \Rightarrow -9a^2 + 9a - 2 = 0 \Rightarrow (-3a+1)(3a+2) = 0 \Rightarrow a = \frac{1}{3} \text{ or } a = \frac{2}{3}$$

4. Calculate $U(f,P_n)$ and $L(f,P_n)$ for the function $f(x) = x$ over the interval $[0,1]$ where P_n is the partition of $[0,1]$ into n subintervals of equal length. Show that $\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n)$, hence f is integrable over $[0,1]$. What is $\int_0^1 f(x)\,dx$?

Solution:



The partition $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}$

$$\Delta x_i = \frac{1-0}{n} = \frac{1}{n}$$

Since f is increasing, to compute upper Riemann sum for each subinterval $[x_{i-1}, x_i]$ choose $x_i^* = x_i = a + i\,\Delta x_i = 0 + \frac{i}{n} = \frac{i}{n} = u_i$ Right end point, to compute lower Riemann sum for each subinterval $[x_{i-1}, x_i]$ choose $x_i^* = x_{i-1} = a + (i-1)\,\Delta x_i = 0 + \frac{i-1}{n} = \frac{i-1}{n} = l_i$ left endpoint.

The lower Riemann sum of f:

$$L(f,P_n) = \sum_{i=1}^{n} f(l_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(\frac{i-1}{n}\right)\cdot\frac{1}{n} = \sum_{i=1}^{n}\left(\frac{i-1}{n}\right)\cdot\frac{1}{n} = \sum_{i=1}^{n}\frac{i}{n^2} - \sum_{i=1}^{n}\frac{1}{n^2} = \frac{1}{n^2}\,\frac{n(n+1)}{2} - n\,.\,\frac{1}{n^2}$$

$$= \frac{n+1}{2n} - \frac{1}{n} = \frac{n-1}{2n}$$

The upper Riemann sum of f:

$$U(f,P_n) = \sum_{i=1}^{n} f(u_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} = \sum_{i=1}^{n}\left(\frac{i}{n}\right)\cdot\frac{1}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{1}{n^2}\,\frac{n(n+1)}{2} = \frac{n^2+n}{2n^2}$$

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty}\frac{n-1}{2n} = \lim_{n\to\infty}\frac{\cancel{n}(1-\overset{0}{1/n})}{2\cancel{n}} = \frac{1}{2}$$

$$\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty}\frac{n^2+n}{2n^2} = \lim_{n\to\infty}\frac{\cancel{n^2}(1+\overset{0}{\tfrac{1}{n}})}{2\cancel{n^2}} = \frac{1}{2}$$

Thus; $\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n)$.

For every n, $L(f,P) \leqslant \lim_{n\to\infty} U(f,P_n) = \frac{1}{2}$ and $U(f,P) \geqslant \lim_{n\to\infty} L(f,P_n) = \frac{1}{2}$. If there exists a number I such that $L(f,P) \leqslant I \leqslant U(f,P)$ for all P, $I = \frac{1}{2}$.

Thus; $\int_0^1 f(x)\,dx = \frac{1}{2}$.

5. Express the following limits as definite integrals

(a) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\ln\left(1+\frac{2i}{n}\right)$

(b) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i^2}$

Solution:

If $f(x)$ is integrable on $[a,b]$, for any partition P_n, $\lim_{n\to\infty} R(f,P,x_i^*) = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = \int_a^b f(x)\,dx$

(a) $$\lim_{n\to\infty}\sum_{i=1}^{n}\underbrace{\frac{2}{n}}_{\Delta x_i}\underbrace{\ln\left(1+\frac{2i}{n}\right)}_{f(x_i^*)}$$

$\Delta x_i = \frac{b-a}{n} = \frac{2}{n} \Rightarrow b-a=2$, Choose $f(x)=\ln x$ which is integrable (continuous) on $\mathbb{R}^+$

$x_i^* = a + i\,\Delta x_i = a + i\cdot\frac{2}{n} = 1+\frac{2i}{n} \Rightarrow a=1\ \&\ b=3$

Thus; $$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\ln\left(1+\frac{2i}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,.\,\Delta x_i = \int_1^3(\ln x)\,dx$$

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2+i^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2\left(1+\left(\frac{i}{n}\right)^2\right)} = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\frac{1}{n}}_{\Delta x_i} \cdot \underbrace{\frac{1}{1+\left(\frac{i}{n}\right)^2}}_{f(x_i^*)}$

$\Delta x_i = \frac{b-a}{n} = \frac{1}{n} \Rightarrow b-a=1$, Choose $f(x) = \frac{1}{1+x^2}$ which is integrable on $\mathbb{R}$.

$x_i^* = \frac{i}{n} = a + i.\Delta x_i = \frac{i}{n} \Rightarrow a=0$ & $b=1$

Thus; $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2+i^2} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*).\Delta x_i = \int_0^1 \frac{1}{1+x^2}\,dx$

6. Let f and g be positive valued continuous functions on $[0,1]$ such that $\int_0^1 f(x)dx = 2$ and $\int_0^1 g(x)\,dx = 3$. Show that $3 \leqslant \int_0^1 \sqrt{(f(x))^2+(g(x))^2}\,dx \leqslant 5$.

Solution:

Observe that $(f(x))^2+(g(x))^2 \leqslant (f(x)+g(x))^2$ so

$\sqrt{(f(x))^2+(g(x))^2} \leqslant \sqrt{(f(x)+g(x))^2} = |f(x)+g(x)| = f(x)+g(x)$ since f & g are positive valued functions.

So $\int_0^1 \sqrt{(f(x))^2+(g(x))^2}\,dx \leqslant \int_0^1 (f(x)+g(x))\,dx = \int_0^1 f(x)dx + \int_0^1 g(x)\,dx = 2+3=5$.

Moreover $\sqrt{(g(x))^2} = g(x) \leqslant \sqrt{(f(x))^2+(g(x))^2}$ so we have

$\int_0^1 g(x)\,dx \leqslant \int_0^1 \sqrt{(f(x))^2+(g(x))^2}\,dx \Rightarrow 3 \leqslant \int_0^1 \sqrt{(f(x))^2+(g(x))^2}\,dx$

Thus; $3 \leqslant \int_0^1 \sqrt{(f(x))^2+(g(x))^2}\,dx \leqslant 5$