Chapter 5: Integration

5.1 Sums and Sigma Notation
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Partition: Let \( P \) be a finite set of points \( P = \{ x_0, x_1, \ldots, x_n \} \) where \( a = x_0 < x_1 < \ldots < x_n = b \). Such a set \( P \) is called a partition of \([a,b]\), divides \([a,b]\) into \( n \) subintervals of which the \( i \)-th is \([x_{i-1}, x_i]\).

The length of the \( i \)-th subinterval is \( \Delta x_i = x_i - x_{i-1} \), \( i \in \{1, \ldots, n\} \).

The norm of the partition \( P \) is \( \| P \| = \max_{1 \leq i \leq n} \Delta x_i \).

Riemann Sum: Let \( f(x) \) be a continuous function defined on \([a,b]\) and \( P = \{ x_0, x_1, \ldots, x_n \} \) be a partition of \([a,b]\).

For arbitrary points on each subinterval, \( x_i^* \in [x_{i-1}, x_i] \), the Riemann Sum of \( f \) corresponding the partition \( P \)

\[
R(f, P, x_i^*) = \sum_{i=1}^{n} f(x_i^*) \Delta x_i = f(x_1^*) \Delta x_1 + f(x_2^*) \Delta x_2 + \ldots + f(x_n^*) \Delta x_n.
\]

Since \( f \) is continuous on \([a,b]\), on each subinterval \([x_{i-1}, x_i]\) it has absolute maximum and minimum values. Thus, there are numbers \( u_i \) and \( v_i \) in \([x_{i-1}, x_i]\) such that

\[
f(u_i) \leq f(x) \leq f(v_i) \quad \text{whenever} \quad x_{i-1} \leq x \leq x_i.
\]

The Lower Riemann Sum of \( f \) : \( \text{L}(f, P) = \sum_{i=1}^{n} f(u_i) \Delta x_i = f(u_1) \Delta x_1 + f(u_2) \Delta x_2 + \ldots + f(u_n) \Delta x_n \)

The Upper Riemann Sum of \( f \) : \( \text{U}(f, P) = \sum_{i=1}^{n} f(v_i) \Delta x_i = f(v_1) \Delta x_1 + f(v_2) \Delta x_2 + \ldots + f(v_n) \Delta x_n \).
**The Definite Integral:** Suppose there is exactly one number $I$ such that for every partition $P$ of $[a,b]$ we have $|L(P) - I|$ and $|U(P) - I|$. Then we say that the function $f$ is integrable on $[a,b]$, $I$ is the definite integral of $f$ on $[a,b]$.

$$I = \int_{a}^{b} f(x) \, dx$$

If $f$ is integrable on $[a,b]$, then

$$\lim_{n \to \infty} \lim_{\pi \to 0} R(f, P, c) = \int_{a}^{b} f(x) \, dx.$$

1. Compute the lower and upper Riemann sum of the function $f(x) = \sin(\pi x)$ over the interval $[0,1]$ using the partition of $[0,1]$ into four subintervals of equal length.

**Solution:**

Sketch the graph of the function $f(x) = \sin(\pi x)$ over $[0,1]$.

To compute lower Riemann sum, we need to find absolute minimum value of $f$ on each subinterval. On $[0, \frac{1}{4}]$, $f(0)$; on $[\frac{1}{4}, \frac{1}{2}]$, $f(\frac{1}{4})$; on $[\frac{1}{2}, \frac{3}{4}]$, $f(\frac{3}{4})$ and on $[\frac{3}{4}, 1]$, $f(1)$ are absolute minimum values of $f$. Thus the lower Riemann sum of $f$ corresponding $P$ is

$$L(P) = \sum_{i=1}^{4} f(i) \Delta x_i = f(0) \Delta x_1 + f \left( \frac{1}{4} \right) \Delta x_2 + f \left( \frac{3}{4} \right) \Delta x_3 + f(1) \Delta x_4$$

$$= \frac{1}{4} \left( f(0) + f \left( \frac{1}{4} \right) + f \left( \frac{3}{4} \right) + f(1) \right) = \frac{1}{4} \left( 0 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 0 \right) = \frac{\sqrt{2}}{4}$$

To compute upper Riemann sum, we need to find absolute maximum value of $f$ on each subinterval. On $[0, \frac{1}{4}]$, $f(\frac{1}{4})$; on $[\frac{1}{4}, \frac{1}{2}]$, $f(\frac{1}{4})$; on $[\frac{1}{2}, \frac{3}{4}]$, $f(\frac{3}{4})$ and on $[\frac{3}{4}, 1]$, $f(\frac{3}{4})$ are absolute maximum values of $f$. Thus the upper Riemann sum of $f$ corresponding $P$ is

$$U(P) = \sum_{i=1}^{4} f(i) \Delta x_i = f \left( \frac{1}{4} \right) \Delta x_1 + f \left( \frac{3}{4} \right) \Delta x_2 + f \left( \frac{3}{4} \right) \Delta x_3 + f(1) \Delta x_4$$

$$= \frac{1}{4} \left( f \left( \frac{1}{4} \right) + f \left( \frac{3}{4} \right) + f \left( \frac{3}{4} \right) + f(1) \right) = \frac{1}{4} \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$$
2. Show that for any continuous increasing function \( f(x) \) defined on \( \mathbb{R} \) such that \( f(0) = 0, f(1/4) = 2, f(1/2) = 4, f(3/4) = 50 \) and \( f(1) = 120 \), we have that
\[
14 < \int_0^1 f(x) \, dx < 44
\]

**Solution:**

Since \( f \) is a continuous function, it is integrable on \([0,1]\).

Moreover, for every partition of \([0,1]\) we have the following inequality since \( f \) is integrable.

\[
\underline{L}(f,P) < \int_0^1 f(x) \, dx < \overline{U}(f,P)
\]

Let \( P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\} \) be a partition for \([0,1]\) and each subinterval has length

\[
\Delta x_i = \frac{1 - 0}{4} = \frac{1}{4}
\]

To write lower Riemann sum, since \( f \) is increasing we need to choose left end points of subintervals:

\[
\underline{L}(f,P) = \sum_{i=1}^4 f(x_i) \Delta x_i = f(0) \Delta x_1 + f\left(\frac{1}{4}\right) \Delta x_2 + f\left(\frac{1}{2}\right) \Delta x_3 + f(1) \Delta x_4
\]

\[
= \frac{1}{4} \left( 0 + 2 + 4 + 50 \right) = 14
\]

To write upper Riemann sum, since \( f \) is increasing we need to choose right end points of subintervals:

\[
\overline{U}(f,P) = \sum_{i=1}^4 f(x_i) \Delta x_i = f\left(\frac{1}{4}\right) \Delta x_1 + f\left(\frac{1}{2}\right) \Delta x_2 + f\left(\frac{3}{4}\right) \Delta x_3 + f(1) \Delta x_4
\]

\[
= \frac{1}{4} \left( 2 + 4 + 50 + 120 \right) = 44
\]

Therefore,

\[
\underline{L}(f,P) < \int_0^1 f(x) \, dx < \overline{U}(f,P) \Rightarrow 14 < \int_0^1 f(x) \, dx < 44
\]
3. Find a partition of \([1,1]\) into two subintervals such that the lower Riemann sum for
the function \(f(x) = 1x1\) over \([1,1]\) corresponding to the partition is \(\frac{2}{9}\).

**Solution:**

Consider the partition \(P = \{-1, a, 1\}\) and assume that \(a > 0\).

On \([-1, a]\), \(f(x)\) is absolute minimum value of \(f\) and
on \([a, 1]\), \(f(x)\) is absolute minimum value of \(f\).

The lower Riemann sum of \(f\) corresponding to \(P\) is

\[
L(f, P) = \sum_{i=1}^{2} f(x_i) \Delta x_i = f(0) \Delta x_1 + f(0) \Delta x_2
\]

\[
= 0 \cdot (a - (-1)) + |a| \cdot (1 - a) \quad \text{(since } a > 0, \ |a| = a)\]

\[
= a - a^2
\]

\[
a - a^2 = \frac{2}{9} \Rightarrow -9a^2 + 9a - 2 = 0 \Rightarrow (-3a + 1)(3a + 2) = 0 \Rightarrow a = \frac{1}{3} \text{ or } a = \frac{2}{3}
\]

4. Calculate \(L(f, P_n)\) and \(U(f, P_n)\) for the function \(f(x) = x\) over the interval \([0,1]\) where
\(P_n\) is the partition of \([0,1]\) into \(n\) subintervals of equal length. Show that
\[
\lim_{n \to \infty} L(f, P_n) = \lim_{n \to \infty} U(f, P_n), \text{ hence } f \text{ is integrable over } [0,1]. \text{ What is } \int_0^1 f(x) \, dx ?
\]

**Solution:**

The partition \(P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}\)

\[
\Delta x_i = \frac{1 - 0}{n} = \frac{1}{n}
\]

Since \(f\) is increasing, to compute upper Riemann sum for
each subinterval \([x_{i-1}, x_i]\) choose \(x_i^{*} = x_i = a + i \Delta x_i = 0 + i \cdot \frac{1}{n} = i \cdot \frac{1}{n}\)
right endpoint, to compute lower Riemann sum for each
subinterval \([x_{i-1}, x_i]\) choose \(x_i^{*} = x_{i-1} = a + (i-1) \Delta x_i = 0 + (i-1) \cdot \frac{1}{n} = i-1 \cdot \frac{1}{n}\)
left endpoint.

The lower Riemann sum of \(f\):

\[
L(f, P_n) = \sum_{i=1}^{n} f(x_i) \Delta x_i = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right) \cdot \frac{1}{n} = \sum_{i=1}^{n} \frac{i-1}{n^2} - \sum_{i=1}^{n} \frac{1}{n^2} = \frac{1}{n^2} \cdot n(n+1) - n \cdot \frac{1}{n^2} = \frac{n+1}{2n} - \frac{n-1}{2n}
\]

\[
= \frac{n+1 - n}{2n} = \frac{n-1}{2n}
\]
The upper Riemann sum of $f$.

\[ U(f,P_n) = \sum_{i=1}^{n} f(u_i) \Delta x_i = \sum_{i=1}^{n} f \left( \frac{i}{n} \right) \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n^2 + n}{2n} \]

\[ \lim_{n \to \infty} L(f,P_n) = \lim_{n \to \infty} \frac{n-1}{2n} = \lim_{n \to \infty} \frac{n(1 - \frac{1}{n})}{2} = \frac{1}{2} \]

Thus, \( \lim_{n \to \infty} U(f,P_n) = \lim_{n \to \infty} L(f,P_n) \).

\[ \lim_{n \to \infty} U(f,P_n) = \lim_{n \to \infty} \frac{n^2 + n}{2n^2} = \lim_{n \to \infty} \frac{n^2 (1 + \frac{1}{n})}{2n} = \frac{1}{2} \]

For every $n$, \( L(f,P_n) \leq \lim_{n \to \infty} U(f,P_n) = \frac{1}{2} \) and \( U(f,P_n) \geq \lim_{n \to \infty} L(f,P_n) = \frac{1}{2} \). Thus, \( \int_0^1 f(x) dx = \frac{1}{2} \).

5. Express the following limits as definite integrals

(a) \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n} \ln \left( 1 + \frac{2i}{n} \right) \)

(b) \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{n}{n^2 + i^2} \)

Solution:

If $f(x)$ is integrable on $[a,b]$, for any partition $P_n$, \( \lim_{n \to \infty} R(f,P_n,x_i) = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)^* \Delta x_i = \int_a^b f(x) dx \)

(a) \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n} \ln \left( 1 + \frac{2i}{n} \right) \)

\( \Delta x_i = b - a = 2 \), Choose \( f(x) = \ln x \) which is integrable (continuous) on $R^+$

\( x_i^* = a + i \Delta x_i = a + i \frac{2}{n} = 1 + \frac{2i}{n} \) \( \Rightarrow a = 1 \) \( \& \) \( b = 3 \)

Thus, \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n} \ln \left( 1 + \frac{2i}{n} \right) = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x_i = \int_1^3 (\ln x) dx \)

\( \int_1^3 (\ln x) dx = \left[ x \ln x - x \right]_1^3 = 3 \ln 3 - 3 - (1 \ln 1 - 1) = 3 \ln 3 - 2 \)

\[ \int_0^1 f(x) dx = \frac{1}{2} \]

\[ \int_1^3 (\ln x) dx = 3 \ln 3 - 2 \]
(b) \[ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{n}{n^2 + i^2} = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n^2 \left(1 + \frac{i^2}{n^2}\right)} = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{\left(\frac{i}{n}\right)^2} \Delta x_i \cdot f\left(\frac{i}{n}\right) \]

\[ \Delta x_i = \frac{b-a}{n} = \frac{1}{n} \Rightarrow b-a = 1 \]

Choose \( f(x) = \frac{1}{1+x^2} \) which is integrable on \( \mathbb{R} \).

\[ x_i = \frac{i}{n} = a + \Delta x_i = \frac{i}{n} \Rightarrow a = 0 \text{ and } b = 1 \]

Thus, \[ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{n}{n^2 + i^2} = \int_{0}^{1} f(x) \Delta x_i = \int_{0}^{1} \frac{1}{1+x^2} \, dx \]

6. Let \( f \) and \( g \) be positive valued continuous functions on \([0,1]\) such that \[ \int_{0}^{1} f(x) \, dx = 2 \]

and \[ \int_{0}^{1} g(x) \, dx = 3 \]. Show that \( 3 \leq \int_{0}^{1} \sqrt{f(x)^2 + g(x)^2} \, dx \leq 5 \).

**Solution:**

Observe that \( (f(x))^2 + (g(x))^2 \leq (f(x) + g(x))^2 \) so

\[ \sqrt{(f(x))^2 + (g(x))^2} \leq \sqrt{(f(x) + g(x))^2} = |f(x) + g(x)| = f(x) + g(x) \]

Since \( f \) \& \( g \) are positive valued functions.

So \[ \int_{0}^{1} \sqrt{(f(x))^2 + (g(x))^2} \, dx \leq \int_{0}^{1} (f(x) + g(x)) \, dx = \int_{0}^{1} f(x) \, dx + \int_{0}^{1} g(x) \, dx = 2 + 3 = 5 \]

Moreover \( \sqrt{(g(x))^2} = g(x) \leq \sqrt{(f(x))^2 + (g(x))^2} \) so we have

\[ \int_{0}^{1} g(x) \, dx \leq \int_{0}^{1} \sqrt{(f(x))^2 + (g(x))^2} \, dx \Rightarrow 3 \leq \int_{0}^{1} \sqrt{(f(x))^2 + (g(x))^2} \, dx \]

Thus, \( 3 \leq \int_{0}^{1} \sqrt{(f(x))^2 + (g(x))^2} \, dx \leq 5 \).