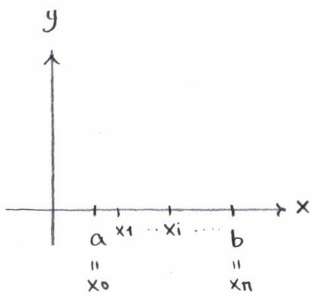
MATH 117 & MATH 119

RECITATION 5

Chapter 5 : Integration

5.1 Sums and Sigma Notation
5.2 Areas as Limits of Sums
5.3 The Definite Integral
5.4 Properties of Definite Integral

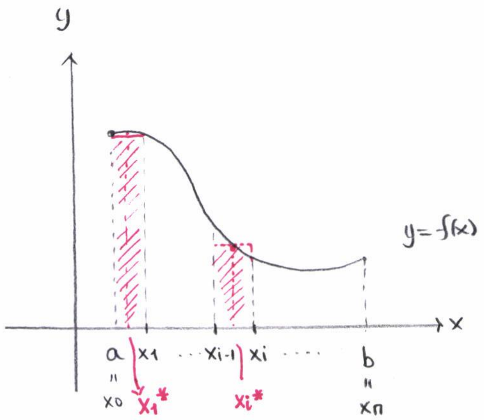
Partition: Let P be a finite set of points $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ where $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$. Such a set P is called a partition of $[a,b]$, divides $[a,b]$ into n subintervals of which the ith is $[x_{i-1}, x_i]$.



The length of ith subinterval is $\Delta x_i = x_{i-1} - x_i$, $i \in \{1, 2, \dots n\}$

The norm of the partition P is $\|P\| = \max\limits_{1 \leq i \leq n} \Delta x_i$

Riemann Sum: Let $f(x)$ be a continuous function defined on $[a,b]$ and $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$.



For arbitrary points on each subintervals, $x_i^* \in [x_{i-1}, x_i]$, the Riemann Sum of f corresponding the partition P

$$R(f, P, x_i^*) = \sum_{i=1}^{n} f(x_i^*) . \Delta x_i$$

$$= f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + \dots + f(x_n^*)\,\Delta x_n .$$

Since f is continuous on $[a,b]$, on each subinterval $[x_{i-1}, x_i]$ it has absolute maximum and minimum values. Thus, there are numbers l_i and u_i in $[x_{i-1}, x_i]$ such that $f(l_i) \leq f(x) \leq f(u_i)$ whenever $x_{i-1} \leq x \leq x_i$.

The Lower Riemann Sum of f : $L(f,P) = f(l_1)\,\Delta x_1 + f(l_2) . \Delta x_2 + \dots + f(l_n)\,\Delta x_n$

$$= \sum_{i=1}^{n} f(l_i)\,\Delta x_i$$

The Upper Riemann Sum of f : $U(f,P) = f(u_1) . \Delta x_1 + f(u_2) . \Delta x_2 + \dots + f(u_n) . \Delta x_n$

$$= \sum_{i=1}^{n} f(u_i) . \Delta x_i$$

The Definite Integral : Suppose there is exactly one number I such that for every partition P of $[a,b]$ we have $L(f,P) \leq I \leq U(f,P)$, then we say that the function f is integrable on $[a,b]$, I is the definite integral of f on $[a,b]$.

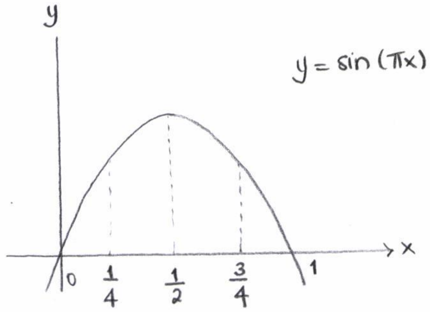
$$I = \int_a^b f(x)\,dx$$

If f is integrable on $[a,b]$, then $\lim\limits_{\substack{n \to \infty \\ \|P\| \to 0}} R(f,P,c) = \int_a^b f(x)\,dx$.

1. Compute the lower and upper Riemann sum of the function $f(x) = \sin(\pi . x)$ over the interval $[0,1]$ using the partition of $[0,1]$ into four subintervals of equal length.

Solution:

Sketch the graph of the function $f(x) = \sin(\pi x)$ over $[0,1]$.



The partition $P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ divides $[0,1]$ into four subintervals of equal length.

$\Delta x_i = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$ for each subinterval.

To compute lower Riemann sum, we need to find absolute minimum value of f on each subinterval. On $[0,\frac{1}{4}]$, $f(0)$; on $[\frac{1}{4},\frac{1}{2}]$, $f(\frac{1}{4})$; on $[\frac{1}{2},\frac{3}{4}]$, $f(\frac{3}{4})$ and on $[\frac{3}{4},1]$, $f(1)$ are absolute minimum values of f. Thus the lower Riemann sum of f corresponding P is

$$L(f,P) = \sum_{i=1}^{4} f(l_i)\,\Delta x_i = f(0)\,\Delta x_1 + f(\tfrac{1}{4})\,\Delta x_2 + f(\tfrac{3}{4})\,\Delta x_3 + f(1)\,\Delta x_4$$

$$= \frac{1}{4}\left(f(0) + f(\tfrac{1}{4}) + f(\tfrac{3}{4}) + f(1)\right) = \frac{1}{4}\left(0 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 0\right) = \frac{\sqrt{2}}{4}$$

To compute upper Riemann sum, we need to find absolute maximum value of f on each subinterval. On $[0,\frac{1}{4}]$, $f(\frac{1}{4})$; on $[\frac{1}{4},\frac{1}{2}]$, $f(\frac{1}{2})$; on $[\frac{1}{2},\frac{3}{4}]$, $f(\frac{1}{2})$ and on $[\frac{3}{4},1]$, $f(\frac{3}{4})$ are absolute maximum values of f. Thus the upper Riemann sum of f corresponding P is

$$U(f,P) = \sum_{i=1}^{4} f(u_i)\,\Delta x_i = f(\tfrac{1}{4})\,\Delta x_1 + f(\tfrac{1}{2})\,\Delta x_2 + f(\tfrac{1}{2})\,\Delta x_3 + f(\tfrac{3}{4})\,\Delta x_4$$

$$= \frac{1}{4}\left(f(\tfrac{1}{4}) + f(\tfrac{1}{2}) + f(\tfrac{1}{2}) + f(\tfrac{3}{4})\right) = \frac{1}{4}\left(\frac{\sqrt{2}}{2} + 1 + 1 + \frac{\sqrt{2}}{2}\right) = \frac{1}{2} + \frac{\sqrt{2}}{4}$$