M E T U Department of Mathematics

Math 116 Basic Algebraic Structures Spring 2019 Midterm III 7 May 2019 17:40		
FULL NAME	STUDENT ID	DURATION 80 MINUTES
4 QUESTIONS ON 2 PAGES		TOTAL 40 POINTS

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

(2+2+2+2 pts) **1.** Consider the permutation $f = (1372)(7536)(52439)(829) \in S_{12}$

a) Write f in permutation notation, $f = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 7 & 9 & 6 & 1 & 2 & 5 & 4 & 8 & 10 & 11 & 12 \end{bmatrix}$

b) Write f as a product of disjoint cycles.

$f = (139846275)$

c) Is $f \in A_{12}$? Explain.

f is a cycle of length 9 and hence is an even permutation. Therefore $f \in A_{12}$.

d) Find the order of f.

f is a cycle of length 9 and hence has order 9.

(3+4+4 pts) **2.**

a) Let G be a group and H a subgroup of G. Complete the following definition:

Definition: H is a normal subgroup of G if for any $g \in G$, we have $gH = Hg$.

b) Let $H = \{e, (12)(34)\} \subseteq S_4$. Is H a normal subgroup of S_4? Explain.

Note that if we let $\sigma = (13) \in S_4$, then $\sigma(12)(34)\sigma^{-1} = (32)(14) \notin H$. Hence H is NOT a normal subgroup.

c) Prove the following: If H and K are normal subgroups of a group G, then $H \cap K$ is a subgroup of G and is normal.

Suppose H and K are normal subgroups of G. We will first show that $H \cap K$ is a subgroup of G. Since $e \in H \cap K$, $H \cap K \neq \emptyset$. Let $x, y \in H \cap K$. Then $x, y \in H \implies xy^{-1} \in H$ and $x, y \in K \implies xy^{-1} \in K$. Therefore $xy^{-1} \in H \cap K$. Thus $H \cap K$ is a subgroup of G.

Now, let $x \in H \cap K$ and $g \in G$. $x \in H \implies gxg^{-1} \in H$, since H is a normal subgroup of G. Similarly, $gxg^{-1} \in K$. Therefore $gxg^{-1} \in H \cap K$, which shows that $H \cap K$ is a normal subgroup of G.

(3+3+3 pts) **3.** Consider the subgroups $4\mathbb{Z}$ and $16\mathbb{Z}$ of the group $\mathbb{Z}$. You are given that $16\mathbb{Z}$ is a normal subgroup of $4\mathbb{Z}$.

a) List the elements of the quotient group $4\mathbb{Z}/16\mathbb{Z}$.

$4\mathbb{Z}/16\mathbb{Z} = \{16\mathbb{Z}, 4+16\mathbb{Z}, 8+16\mathbb{Z}, 12+16\mathbb{Z}\}$

b) Construct the addition table of the group $4\mathbb{Z}/16\mathbb{Z}$ using the table on the right.

$+$	$16\mathbb{Z}$	$4+16\mathbb{Z}$	$8+16\mathbb{Z}$	$12+16\mathbb{Z}$
$16\mathbb{Z}$	$16\mathbb{Z}$	$4+16\mathbb{Z}$	$8+16\mathbb{Z}$	$12+16\mathbb{Z}$
$4+16\mathbb{Z}$	$4+16\mathbb{Z}$	$8+16\mathbb{Z}$	$12+16\mathbb{Z}$	$16\mathbb{Z}$
$8+16\mathbb{Z}$	$8+16\mathbb{Z}$	$12+16\mathbb{Z}$	$16\mathbb{Z}$	$4+16\mathbb{Z}$
$12+16\mathbb{Z}$	$12+16\mathbb{Z}$	$16\mathbb{Z}$	$4+16\mathbb{Z}$	$8+16\mathbb{Z}$

c) Find the order of $12+16\mathbb{Z}$.

$\langle 12+16\mathbb{Z}\rangle = 4\mathbb{Z}/16\mathbb{Z}$ thus $12+16\mathbb{Z}$ has order 4.

(3+3+3+3 pts) **4.** You are given that the set $R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a ring with respect to matrix addition and matrix multiplication.

a) Is R a commutative ring? Explain.

Let $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \in R$.

Then $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} x & y \\ 0 & x \end{bmatrix} = \begin{bmatrix} ax & ay+bx \\ 0 & ax \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. So, R is a commutative ring.

b) Does R have zero divisors? Explain.

Note that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a non-zero element of R. We have $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Thus, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in R$ is a zero divisor.

c) Is R an integral domain? Explain.

R is not an integral domain since it has zero divisors.

d) Is R a field? Explain.

We have proven in class that every field is an integral domain. Hence R is not a field.