5. (15 pts.) Let $f = (1,7,2,6,5)$ and $g = (4,5,3,2)$ be cycles in S_7.

(a) Write the product fg as a product of disjoint cycles.

$f \cdot g = (1\,7\,2\,4)(3\,6\,5)$

(b) Write fg as a product of transpositions.

$fg = (14)(12)(17)(35)(36)$

(c) Determine whether fg is even or odd.

fg is a product of odd number of traspositions.
So fg is odd.

(d) Find the order of fg.

$fg = (1\,7\,2\,4)(3\,6\,5)$

$o((1\,7\,2\,4)) = 4$ and $o((3\,6\,5)) = 3$

$(1\,7\,2\,4)$ and $(3\,6\,5)$ are disjoint cycles.

So $o(fg) = \mathrm{lcm}(o(1\,7\,2\,4), o(3\,6\,5)) = \mathrm{lcm}(4,3) = 12$

(e) Find $(fg)^{75}$.

$75 = 12.6 + 3 \Rightarrow (fg)^{75} = (fg)^{12.6+3} = ((fg)^{12})^6 \cdot (fg)^3$

$(fg)^{12} = (1)$ $\qquad = (1).(1\,4\,2\,7) = (1\,4\,2\,7)$.

M E T U
Department of Mathematics

Basic Algebraic Structures Midterm I	
Code : Math 116 Acad. Year : 2016-2017 Semester : Spring Instructor : G.E., E.C, S.F., F.Ö, E.S.	Last Name : Name : Student No : Department : Section: Signature :
Date : 06.04.2017 Time : 17.40 Duration : 100 minutes	5 Questions on 4 Pages Total 60 Points

1	2	3	4	5	

READ THE PROBLEMS CAREFULLY AND GIVE DETAILED WORK

1.(10 pts.) Find all incongruent solutions of the congruence $6x \equiv 15 \pmod{27}$.

$6x \equiv 15 \mod 27 \iff 27 \mid (6x-15)$

$\iff 27 \mid 3(2x-5)$

$\iff 9 \mid (2x-5)$

$\iff 2x \equiv 5 \pmod 9$

$\iff 2.5 \equiv 5.5 \mod 9$

as $(5,9)=1$

$\iff x \equiv 25 \mod 9$

$\iff x \equiv 7 \mod 9$

Hence $x \in \{\underline{7} + 27\mathbb{Z}, \underline{7+9} + 27\mathbb{Z}, \underline{7+9.2} + 27\mathbb{Z}\}$

All incongruent solutions mod 27 are

$\{7, 7+9, 7+9.2\} = \{7, 16, 25\}$.

2.(5 pts.) Let H be a subgroup of the group G. Prove that $gHg^{-1} = \{ghg^{-1} : h \in H\}$ is a subgroup of G for any $g \in G$.

(1) Since $e = geg^{-1} \in gHg^{-1}$, $gHg^{-1} \neq \emptyset$

(2) Let gh_1g^{-1}, gh_2g^{-1} be two arbitrary elements in gHg^{-1} for some $h_1, h_2 \in H$. Then

$$(gh_1g^{-1})(gh_2g^{-1}) = gh_1\underbrace{(g^{-1}g)}_{e}h_2g^{-1} = g\underbrace{h_3}_{\in H}g^{-1} \in gHg^{-1}$$

gHg^{-1} is closed under $\cdot$.

(3) Let $x = ghg^{-1}$ be an arbitrary element in gHg^{-1}

Then $x^{-1} = (ghg^{-1})^{-1} = gh^{-1}g^{-1} \in gHg^{-1}$. Thus, $gHg^{-1} \leq G$ $\forall g \in G$.

3. (10 pts.) Let G be a group and $f : G \longrightarrow G$ be the map defined by $f(x) = x^{-1}$. Prove that f is an isomorphism if and only if G is abelian.

f is 1-1: $x^{-1} = y^{-1} \Rightarrow (x^{-1})^{-1} = (y^{-1})^{-1} \Rightarrow x = y \quad \forall x, y \in G$.

f is onto: $f(x^{-1}) = (x^{-1})^{-1} = x \quad \forall x \in G$

f is a group homomorphism $\Rightarrow$ G is abelian:

For all $g_1, g_2 \in G$, $g_i = f(x_i)$ for some $x_i \in G$ as f is onto.

$$g_1g_2 = f(x_1).f(x_2) \underset{\text{f is a hom.}}{=} f(x_1x_2) = (x_1x_2)^{-1} = x_2^{-1}x_1^{-1} = g_2g_1$$

G is abelian $\Rightarrow$ f is a homomorphism

$$f(x_1x_2) = (x_1x_2)^{-1} = x_2^{-1}x_1^{-1} \underset{\text{G is abelian}}{=} x_1^{-1}x_2^{-1} = f(x_1)f(x_2) \quad \forall x_1, x_2 \in G.$$

4. (20 pts.) Let $G = \langle a \rangle$ where $|G| = 15$. $\quad G = \{a^0 = e, a^1, a^2, \ldots, a^{14}\}$

(a) List all elements that generate G.

a^k, $(k, 15) = 1$

$a, a^2, a^4, a^7, a^8, a^{11}, a^{13}, a^{14}$ generate G.

(b) List all subgroups of G.

$\langle e \rangle = \{e = a^0\}$. trivial subgroup

$\langle a^3 \rangle = \{e, a^3, a^6, a^9, a^{12}\}$

$\langle a^5 \rangle = \{e, a^5, a^{10}\}$

$\langle a \rangle = G$

4 subgroups

(c) List all elements of order 5.

$\langle a^3 \rangle = \langle a^6 \rangle = \langle a^9 \rangle = \langle a^{12} \rangle = \{e, a^3, a^6, a^9, a^{12}\}$

subgroup of order 5.

$\Rightarrow a^3, a^6, a^9, a^{12}$ are elements of order 5.

(d) Are there any elements of order 4 in G? Why?

$o(x) \mid o(G)$

So, G may have elements of order 1, 3, 5, 15. but no elements of order 4 since $4 \nmid 15$.