SECOND MIDTERM

6th May 2016. Duration : 100 minutes.

Three questions : $(12 + 5 + 3) + 10 + 10$, $20 + 10 + 5$, $7 + (5 + 3) + 10$

STUDENT NUMBER	NAME, FAMILY NAME	Q1	Q2	Q3	TOTAL

1. (A) Show [1] that the set

$$R = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \,\middle|\, x, y, z \in \mathbb{Z} \right\}$$

of 2×2 matrices constitute a ring under the usual addition and multiplication of matrices. Show that R is <u>not</u> a commutative ring but it has a multiplicative identity.

(B) Prove that the set

$$I = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \,\middle|\, x, y \in \mathbb{Z} \right\}$$

is an ideal in R.

(C) Is

$$J = \left\{ \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \,\middle|\, x, y \in \mathbb{Z} \right\}$$

an ideal in R?

(A) The matrix addition is associative & commutative with the zero matrix as the identity element. The matrix multiplication is associative and distributive over addition. Thus, to prove that R is a ring it is sufficient to observe the following: (1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in R$ (2) R is closed under addition & multiplication : $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} + \begin{bmatrix} x' & y' \\ 0 & z' \end{bmatrix} = \begin{bmatrix} x+x' & y+y' \\ 0 & z+z' \end{bmatrix} \in R$, $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} x' & y' \\ 0 & z' \end{bmatrix} = \begin{bmatrix} xx' & xy'+yz' \\ 0 & zz' \end{bmatrix} \in R$

for any $x, y, z, x', y', z' \in \mathbb{Z}$. (3) $-\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} -x & -y \\ 0 & -z \end{bmatrix} \in R$ for any $x, y, z \in \mathbb{Z}$.

Clearly $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in R$ is the multiplicative identity. On the other hand R is <u>not</u> commutative since

$$\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$ for instance.

(B) Again, to show that I is a <u>subring</u> it is sufficient to note that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in I$ and $\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} x' & y' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x+x' & y+y' \\ 0 & 0 \end{bmatrix} \in I$ and $\begin{bmatrix} -x & -y \\ 0 & 0 \end{bmatrix} \in I$ for any $x, y, x', y' \in \mathbb{Z}$.

Moreover $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ax & ay \\ 0 & 0 \end{bmatrix} \in I$ & $\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} xa & xb+yc \\ 0 & 0 \end{bmatrix} \in I$

for all $a, b, c, x, y \in \mathbb{Z}$. Hence I is an <u>ideal</u>.

---

[1] It may be assumed that matrix addition is associative and commutative and admits the zero matrix as identity and that the matrix multiplication is associative and distributive over addition.

(C) J is <u>not</u> an ideal in R. The fact is, it is <u>not even</u> in R !

2. Consider the binary operations $\oplus$ and $\odot$ on $\mathbb{Z}$ defined for each $x, y \in \mathbb{Z}$ by

$$x \oplus y = x + y - 1 \quad \text{and} \quad x \odot y = x + y - xy$$

(A) Prove that $\mathbb{Z}$ constitutes an integral domain with respect to the binary operations $\oplus$ and $\odot$.

(B) Let $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$ be the map defined for each $x \in \mathbb{Z}$ by

$$\varphi(x) = 1 - x.$$

Prove that φ is an isomorphism from the ring on $\mathbb{Z}$ with respect to $\oplus$ and $\odot$ to the ring on $\mathbb{Z}$ with the ordinary addition and multiplication.

(C) Is $\mathbb{Z}$ with respect to $\oplus$ and $\odot$ a field?

(A) $\oplus$ is associative : $(x \oplus y) \oplus z = (x+y-1)+z-1 = x+(y+z-1)-1 = x \oplus (y \oplus z)$

" is commutative : $x \oplus y = x+y-1 = y+x-1 = y \oplus x$

" admits 1 as identity : $x \oplus 1 = x+1-1 = x$

$2-x$ is the additive inverse of $x \in \mathbb{Z}$: $x \oplus (2-x) = x+2-x-1 = 1$
$\in \mathbb{Z}$

$\odot$ is associative $(x \odot y) \odot z = (x+y-xy)+z - (x+y-xy)z$
$= x+y+z-yz-zx-xy+xyz$
$= x+(y+z-yz) - x(y+z-yz) = x \odot (y \odot z)$

, is commutative :

$x \odot y = x+y-xy = y+x-yx = y \odot x$

" admits 0 as the multiplicative identity : $x \odot 0 = x+0-x\cdot 0 = x$

$\bullet$" is distributive over $\oplus$:
$x \odot (y \oplus z) = x+(y+z-1)-x(y+z-1) = (x+y-xy)+(x+z-xz) -1 = (x \odot y) \oplus (x \odot z)$

No zero divisors : If $x \neq 1$ & $x \odot y = x+y-xy = 1$ then $y(1-x) = \frac{1-x}{} \neq 0$

hence $y = 1$.

(B) $y = \cancel{x+}\, 1-x$ has a unique solution $x \overset{\in \mathbb{Z}}{}$ for each given $y \in \mathbb{Z}$.
Therefore φ is bijective. Moreover for any $x, y \in \mathbb{Z}$

$\varphi(x \oplus y) = \varphi(x+y-1) = 1-(x+y-1) = (1-x)+(1-y) = \varphi(x) + \varphi(y)$

$\varphi(x \odot y) = \varphi(x+y-xy) = 1-(x+y-xy) = (1-x)(1-y) = \varphi(x)\,\varphi(y)$.

(C) No ! If $\mathbb{Z}$ were a field w.r.t. $\oplus$ & $\odot$ then so would be $\mathbb{Z}$ with its ordinary $+$ & $\cdot$!

3. Consider

$$\alpha = (1\,2\,3\,4) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} \in S_4 \;,\; \beta = (1\,4)(2\,3) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \in S_4$$

where S_4 is the of permutations of the set $\{1,2,3,4\}$. Notice that $\alpha\beta = \beta\alpha^3$, $\alpha^2\beta = \beta\alpha^2$, $\alpha^3\beta = \beta\alpha$ hence

$$G = \left\{ I, \alpha, \alpha^2, \alpha^3, \beta, \alpha\beta, \alpha^2\beta, \alpha^3\beta \right\}$$

is a group under the binary operation of composition of permutations. By a simple inspection it is seen that $H = \left\{ I, \beta \right\}$ and $K = \left\{ I, \alpha^2, \beta, \alpha^2\beta \right\}$ are subgroups of G.

(A) Prove K is a normal subgroup of G.
(B) Prove that H is <u>not</u> a normal subgroup of G. What is the index of H in G?
(C) Show that $\xi^2 \in K$ for any $\xi \in G$.

(A) Check that all left cosets of K are also right cosets.
  <u>Alternatively</u>, notice that K is a group of index 2 hence, normal.

(B) $\{\alpha, \beta\alpha\} = H\alpha \neq \alpha H = \{\alpha, \alpha\beta\}$. Therefore H is not normal

(C) K has index 2. Therefore G/K contains 2 elements.
  It follows that for any $\xi \in G$, $(\xi K)^2 = \xi^2 K = K$
  hence $\xi^2 \in K$.