Question 5. Let G be a group. Define

$$Z(G) = \{a \in G \mid ag = ga \quad \text{for all} \quad g \in G\}.$$

i) (10 pts) Show that $Z(G)$ is a normal subgroup of G.

$Z(G)$ is a subgroup of G:

- $ex = xe \quad \forall x \in G \Rightarrow e \in Z(G)$
- let $x, y \in Z(G) \Rightarrow xg = gx,\ yg = gy \quad \forall g \in G$
  $xyg \overset{y \in Z(G)}{=} xgy \overset{x \in Z(G)}{=} gxy \Rightarrow xy \in Z(G)$
- $x \in Z(G) \Rightarrow xg = gx \quad \forall g \in G$
  $x^{-1}xgx^{-1} = x^{-1}gxx^{-1} \Rightarrow gx^{-1} = x^{-1}g \quad \forall g \in G$
  $\Rightarrow x^{-1} \in Z(G)$

$Z(G)$ is normal in G:

let $g \in G,\ h \in Z(G)$

$ghg^{-1} \overset{h \in Z(G)}{=} g(g^{-1}h) = h \in Z(G)$

$\Rightarrow Z(G)$ is a normal subgroup of G.

ii) (10 pts) Prove or disprove: If $\varphi : G \to H$ is a group isomorphism between the groups G and H, then $\varphi(Z(G)) = Z(H)$.

$(\subseteq)$ Let $\varphi(x) \in \varphi(Z(G))$ and $h \in H$

Since φ is an isom. $\exists g \in G$ s.t. $\varphi(g) = h$

$\varphi(x) \cdot h = \varphi(x) \cdot \varphi(g) = \varphi(xg) \overset{x \in Z(G)}{=} \varphi(gx) = \varphi(g) \cdot \varphi(x) = h \cdot \varphi(x)$

Since this is true for all $h \in H$

$\varphi(x) \in Z(G)$

$\Rightarrow \varphi(Z(G)) \subseteq Z(H)$

$(\supseteq)$ Let $c \in Z(H)$. Since φ is an isom.

$\exists d \in G$ s.t. $\varphi(d) = c$

Let $g \in G \Rightarrow \varphi(g) \in H \quad \varphi(g) \cdot c \overset{c \in Z(H)}{=} c \cdot \varphi(g)$

$\Rightarrow \varphi(g) \cdot \varphi(d) = \varphi(gd) \overset{c = \varphi(d) \in Z(H)}{=} \varphi(dg) = \varphi(d) \cdot \varphi(g)$

$\overset{\varphi \text{ is } 1\text{-}1}{\Rightarrow} gd = dg$

Since this is true for all $g \in G$

$\Rightarrow d \in Z(G) \Rightarrow c \in \varphi(Z(G))$

$\Rightarrow Z(H) \subseteq \varphi(Z(G))$

BONUS (10 points) Let G be a group with $Z(G) = C$. If the quotient group G/C is cyclic, then G is abelian (commutative).

$C = Z(G) = \{a \in G \mid ag = ga \text{ for all } g \in G\}$

Suppose aC is the generator of G/C.

For $b, d \in G \quad bC, dC \in G/C$

$\Rightarrow bC = a^n C,\ dC = a^m C$ for some $n, m \in \mathbb{Z}$

$\Rightarrow b = a^n z_1,\ d = a^m z_2$ for some $z_1, z_2 \in C$

$bc = a^n z_1 a^m z_2 \overset{z_1 \in C}{=} a^{n+m} z_1 z_2 \overset{z_2 \in C}{=} a^{m+n} z_2 z_1$

$\overset{z_2 \in C}{=} a^m z_2 \cdot a^n z_1 = cb$

$\Rightarrow G$ is abelian.

METU
Department of Mathematics

Group	BASIC ALGEBRAIC STRUCTURES MidTerm 2	List No.

Code : *Math 116* Acad. Year : *2009-2010* Semester : *Spring* Coordinator: *M.B.,G.E.,S.P,E.S.*	Last Name : Name : Student No. : Department : Section : Signature :
Date : *May.20.2010* Time : *17:40* Duration : *100 minutes*	5 QUESTIONS ON 4 PAGES TOTAL 100(+10) POINTS

1	2	3	4	5	SHOW YOUR WORK	

Question 1. (20 points) List the cosets of $\langle[7]\rangle$ in the multiplicative group G of $\mathbf{Z}_{16}$. Make the multiplication table for the quotient group $G/\langle[7]\rangle$.

$G = \{[1], [3], [5], [7], [9], [11], [13], [15]\}$

is the multiplicative group of $\mathbb{Z}_{16}$.

$H = \langle [7] \rangle = \{[1], [7]\}$

The cosets of H in G are;

$H = [7]H$

$[3]H = \{[3], [5]\} = [5]H$

$[9]H = \{[9], [15]\} = [15]H$

$[11]H = \{[11], [13]\} = [13]H$

$G/H = \{H, [3]H, [9]H, [11]H\}$

The multiplication table for G/H is:

	H	[3]H	[9]H	[11]H
H	H	[3]H	[9]H	[11]H
[3]H	[3]H	[9]H	[11]H	H
[9]H	[9]H	[11]H	H	[3]H
[11]H	[11]H	H	[3]H	[9]H

Question 2. (20 points) Prove that every finite integral domain is a field.

Let D be a finite integral domain, say $D = \{a_1, \dots, a_n\}$

Since an integral domain is a commutative ring with unity without zero divisors it is enough to show that every element of D is a unit. Let $d \in D$ be a nonzero element. Consider the elements $da_1, \dots, da_n$. These must be distinct since if $da_i = da_j$ for $i \neq j$ $\Rightarrow$ $d(a_i - a_j) = 0$ since $d \neq 0$ $a_i = a_j$.

$da_k = 1$ for some $1 \leq k \leq n$. Since D is commutative also

$a_k d = 1$

$\Rightarrow$ d has a multiplicative inverse a_k in D.

Question 3. (20 points) Let $\phi : R \to S$ be a ring homomorphism between the rings R and S. Let J be an ideal of S. Show that $\phi^{-1}(J) = \{x \in R \mid \phi(x) \in J\}$ is an ideal of R.

- $\phi^{-1}(J)$ is nonempty : since R is a ring $0_R \in R$
  since ϕ is a homom. $\phi(0_R) = 0_J$
  $\Rightarrow$ $0_R \in \phi^{-1}(J)$
- Let $x, y \in R$ $\Rightarrow$ $\phi(x) \in J$ and $\phi(y) \in J$
  Since J is an ideal of S
  $\phi(x) - \phi(y) \in J$
  Since ϕ is an homom. $\phi(x - y) \in J$
  $\Rightarrow$ $x - y \in \phi^{-1}(J)$
- Let $r \in R$ and $x \in \phi^{-1}(J)$
  $\Rightarrow$ $\phi(r) \in S$ and $\phi(x) \in J$
  Since J is an ideal of S $\phi(r)\phi(x) \in J$ and $\phi(x)\phi(r) \in J$
  Since ϕ is an homom. $\phi(rx) \in J$ and $\phi(xr) \in J$
  $\Rightarrow$ $rx \in \phi^{-1}(J)$, $xr \in \phi^{-1}(J)$

$\Rightarrow$ $\phi^{-1}(J)$ is an ideal of R.

Question 4. (20 points) Let $\alpha = (1357)$, $\beta = (2368)$ and $\gamma = (134)(2578)(165)$ be permutations in S_8.

i) Write γ as a product of disjoint cycles.

$\gamma = (1\ 6\ 7\ 8\ 2\ 5\ 3\ 4)$

ii) Find the order of α, β^{103}, γ.

$|\alpha| = 4$

$|\beta| = 4$ $\Rightarrow$ $|\beta^{100}| = 4$ $\Rightarrow$ $|\beta^{103}| = |\beta^3| = |\beta^{-1}|$

$\beta^{-1} = (8\ 6\ 3\ 2)$ $\Rightarrow$ $|\beta^{-1}| = 4$

$|\gamma| = 8$

iii) Determine whether γ is an odd or even permutation.

$\gamma = (1\ 4)(1\ 3)(1\ 5)(1\ 2)(1\ 8)(1\ 7)(1\ 6)$

γ can be written as a product of 7 transpositions

hence γ is an odd permutation.

iv) Find a permutation $\sigma \in S_8$ such that $\sigma\alpha\sigma^{-1} = \beta$.

Two permutations are conjugate iff they have the same cycle structure.

$$\left.\begin{array}{l} \sigma(1) = 2 \\ \sigma(3) = 3 \\ \sigma(5) = 6 \\ \sigma(7) = 8 \end{array}\right\} \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 4 & 6 & 5 & 8 & 7 \end{pmatrix}$$

$\sigma = (1\ 2)(5\ 6)(7\ 8)$.