M E T U
Department of Mathematics

Group	BASIC ALGEBRAIC STRUCTURES Final	List No.

Code : *Math 116* Acad. Year *2009-2010* Semester : *Spring* Instructor *M.B., G.E., S.P, E.S.*	Last Name: Name : Student No: Department Section : Signature :
Date : *06.06.2010* Time : *13:30* Duration : *120 minutes*	5 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5		

Question 1 (20 pts.) For each of the following statements write T or F inside the parenthesis if the statement is true or false respectively.

(1) (T) An element x in a ring R is called idempotent if $x^2 = x$. The only idempotents in an integral domain are 0 and 1. $x(x-1)=0 \rightsquigarrow x=0$ or $x=1$.

(2) (F) Let H and K be subgroups of G then HK is also a subgroup of G. HK is a sbg $\Leftrightarrow$ $HK = KH$

(3) (F) Suppose that a and b are elements of finite order in a group. Then $o(ab) = \text{lcm}(o(a), o(b))$.

(4) (T) Let $m = \text{lcm}(a, b)$ and $d = \gcd(a, b)$ where a and b are positive integers. Then $dm = ab$.

(5) (F) $f(x) = x^4 + 2x^2 + 1$ is an irreducible polynomial over $\mathbb{R}$.

(6) (T) There is no integer solution to the equation $7x \equiv 2(\text{mod}\, 28)$. $55x \equiv 36(\text{mod } 75)$

(7) (T) If D is an integral domain, then $D[x]$ is also an integral domain.

(8) (F) The group $(\mathbb{Z}_{18}, +)$ has exactly five distinct subgroups. 6

(9) (F) $(\mathbb{Z}, +, \cdot)$ is a field.

(10) (T) Every subgroup of a cyclic group is normal.

(6) $ax \equiv b(\text{mod } n)$ has a solution x in integers if $(a,n)=1$.

(8) $|H| \mid 18 \rightsquigarrow |H| = 1, 2, 3, 6, 9, 18$

(10) Every sbg of a cyclic gp is cyclic $\Rightarrow$ abelian $\Rightarrow$ every sbg of an abelian gp is normal.

(6) $\gcd(7, 28) = 7 \Rightarrow 7 \nmid 2$

Question 2 (20 pts.) For the polynomials

$$f(x) = 3x^2 + 2 \quad \text{and} \quad g(x) = x^4 + 5x^2 + 2x + 2 \quad \text{in} \quad \mathbb{Z}_7[x]$$

(a) Find the greatest common divisor $d(x)$ of $f(x)$ and $g(x)$ in $\mathbb{Z}_7[x]$.

$$\begin{array}{l|l} x^4 + 5x^2 + 2x + 2 & 3x^2 + 2 \\ \hline -x^4 \mp 3x^2 & 5x^2 + 3 \\ \hline 2x^2 + 2x + 2 & \\ -2x^2 \mp 6 & \\ \hline 2x + 3 & \end{array}$$

$$\begin{array}{l|l} 3x^2 + 2 & 2x + 3 \\ \hline -3x^2 \mp x & 5x + 3 \\ \hline 2 + 6x & \\ -2 \mp 6x & \end{array}$$

$$g(x) = f(x) \cdot (5x^2+3) + \underbrace{(2x+3)}_{r_1(x)}$$

$$3x^2+2 = r_1(x) \cdot (5x+3) + \underline{\underline{0}}$$

$$r_1(x) = 2x+3$$

$$d(x) = 2^{-1} \cdot r_1(x) = 4r_1(x) = \underline{\underline{x+5}}$$

$$\boxed{d(x) = x+5}$$

(b) Find $s(x)$ and $t(x)$ in $\mathbb{Z}_7[x]$ such that $d(x) = f(x)s(x) + g(x)t(x)$.

$$r_1(x) = g(x) - f(x)\,(5x^2+3)$$

$$d(x) = 4r_1(x) = 4\,g(x) - f(x)\,(6x^2+5)$$

$$= 4\,g(x) + f(x)\,(-6x^2-5)$$

$$= f(x)\,(x^2+2) + g(x)\,4$$

$$\Rightarrow \boxed{s(x) = x^2+2 \,, \quad t(x) = 4}$$

Question 3 (20 pts.) Given $f(x) = x^4 - 6x^3 + 10x^2 + 2x - 15$ over $\mathbb{C}[x]$ with root $2 + i$. Factorize $f(x)$ over $\mathbb{R}$ and $\mathbb{C}$.

$2+i$ is a root $\Rightarrow$ $2-i$ is also a root of $f(x)$

$(2+i)(2-i) = x^2 - 4x + 5$

$$\begin{array}{l|l} \cancel{x^4} - 6x^3 + 10x^2 + 2x - 15 & x^2 - 4x + 5 \\ \hline -\cancel{x^4} \pm 4x^3 \mp 5x^2 & \boxed{x^2 - 2x - 3} \\ \quad -2\cancel{x^3} + 5x^2 + 2x - 15 & \downarrow \\ \quad \pm 2x^3 \mp 8x^2 \pm 10x & (x-3)(x+1) \\ \qquad -3x^2 + 12x - 15 & \\ \qquad \pm 3x^2 \mp 12x \pm 15 & \\ \hline \qquad\quad \underline{0} & \end{array}$$

$f(x) = (x^2 - 4x + 5)(x-3)(x+1)$ in $\mathbb{R}[x]$

$f(x) = (x - (2+i))(x - (2-i))(x-3)(x+1)$ in $\mathbb{C}[x]$

Question 4 (20 pts.)

(a) Let a and b be elements of a finite group G. Prove that a and bab^{-1} have the same order.

Let $o(a) = n$ and $o(bab^{-1}) = m$. Then $a^n = e$ and $(bab^{-1})^m = e$

$(bab^{-1})^m = (bab^{-1})(bab^{-1}) \dots (bab^{-1}) = ba^m b^{-1} = e \Rightarrow a^m = e \Rightarrow n \mid m$

$(bab^{-1})^n = ba^n b^{-1} = beb^{-1} = e \Rightarrow m \mid n$

$m = n$

(b) Assume that H is a cyclic group of order p^2q^2 where p and q are prime numbers. Find the number of all distinct subgroups of H. Give an explanation.

H cyclic gp of finite order m. Then $\forall\, d \mid m$, $\exists$ a uniqe subgroup of G of order d.

$d \mid p^2q^2 \rightsquigarrow d = \underline{1}, p, q, p^2, q^2, pq, p^2q, pq^2, \underline{p^2q^2}$

Question 5 (20 pts.) Let $T_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| a, b, c \in \mathbb{Z} \right\}$ be the ring of all upper triangular matrices over $\mathbb{Z}$.

(a) Prove that $I = \left\{ \begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix} \middle| x, y \in \mathbb{Z} \right\}$ is an ideal of $T_2(\mathbb{Z})$.

1) I is a subring of R: $I \neq \emptyset$ ✓

$m_1 = \begin{bmatrix} 0 & x_1 \\ 0 & y_1 \end{bmatrix}$ and $m_2 = \begin{bmatrix} 0 & x_2 \\ 0 & y_2 \end{bmatrix}$ $\Rightarrow m_1 + m_2 \in I$ ✓

$\Rightarrow -m_1 \in I$ ✓

2) Let $A \in T_2(\mathbb{Z})$ and $B \in I$ then $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and $B = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$

$$AB = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & ax+by \\ 0 & cy \end{pmatrix} \in I$$

$$BA = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & cx \\ 0 & cy \end{pmatrix} \in I$$

I is an ideal of $T_2(\mathbb{Z})$.

(b) Find the quotient ring $T_2(\mathbb{Z})/I$ and construct an isomorphism $\varphi : T_2(\mathbb{Z})/I \longrightarrow \mathbb{Z}$.

$$T_2(\mathbb{Z})/I = \{ m + I \mid m \in T_2(\mathbb{Z}) \} = \left\{ m' + I \,\middle|\, m' = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, a \in \mathbb{Z} \right\}$$

additive quotient group

$$m = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}}_{\in I}$$

$$T_2(\mathbb{Z})/I \longrightarrow \mathbb{Z}$$

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + I \longrightarrow a \quad \text{isomp.}$$