M E T U Department of Mathematics

Math 111 Fundamentals of Mathematics Fall 2018 Midterm IV 27.12.2018 17:40		
Last Name : Name : Student No :	Section : Duration : 65 *minutes*	
4 QUESTIONS ON 2 PAGES		TOTAL 30 POINTS

1	2	3	4			

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature ....

***(4+2+2+4 pts)* 1.** Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

a) Let $f : X \to \mathbb{Z}$ be a function. Define the relation $\sim$ on X by

$$x \sim y \quad \text{if and only if} \quad f(x) = f(y)$$

for all $x, y \in X$. Prove that $\sim$ is an equivalence relation on X.

We will prove that $\sim$ is reflexive, symmetric and transitive.

Let $x \in X$. Then, we have $f(x) = f(x)$ which implies that $x \sim x$. Hence, $\sim$ is reflexive.

Let $x, y \in X$. Assume that $x \sim y$. Then, by definition, $f(x) = f(y)$ and so $f(y) = f(x)$. It follows that $y \sim x$, and hence $\sim$ is symmetric.

Let $x, y, z \in X$. Assume that $x \sim y$ and $y \sim z$. Then, by definition, $f(x) = f(y)$ and $f(y) = f(z)$. It follows that $f(x) = f(z)$ and hence, $x \sim z$. Therefore, $\sim$ is transitive.

b) **For this part of the question only**, assume that $f(x) = \begin{cases} 111 & \text{if } x \text{ is even and } x \neq 5 \\ 115 & \text{if } x \text{ is odd and } x \neq 5 \\ 153 & \text{if } x = 5 \end{cases}$

Find the equivalence class [1].

$[1] = \{x \in X : 1 \sim x\} = \{x \in X : f(1) = f(x)\} = \{x \in X : 115 = f(x)\} = \{1, 3, 7\}$

c) **For this part of the question only**, assume that

$$f(0) = 1 \quad f(1) = 2 \quad f(2) = 1 \quad f(3) = 3 \quad f(4) = 1 \quad f(5) = 2 \quad f(6) = 1 \quad f(7) = 3$$

Find the quotient set $X/\sim$

$X/\sim = \{[x] : x \in X\} = \{[0], [1], [3]\} = \{\{0, 2, 4, 6\}, \{1, 5\}, \{3, 7\}\}$

d) Let $g : X \to \mathbb{Z}$ be an **injective** function. Define the relation $\preccurlyeq$ on X defined by

$$x \preccurlyeq y \quad \text{if and only if} \quad g(x) \leq g(y)$$

for all $x, y \in X$, where $\leq$ denote the **usual** ordering on $\mathbb{Z}$. Prove that $\preccurlyeq$ is a partial ordering on X.

We shall prove that $\preccurlyeq$ is reflexive, anti-symmetric and transitive.

Let $x \in X$. Then, we have $g(x) \leq g(x)$ and so $x \preccurlyeq x$. Thus, $\preccurlyeq$ is reflexive.

Let $x, y \in X$. Assume that $x \preccurlyeq y$ and $y \preccurlyeq x$. Then, by definition, $g(x) \leq g(y)$ and $g(y) \leq g(x)$. It follows that $g(x) = g(y)$. Since g is injective, this implies that $x = y$. Thus $\preccurlyeq$ is anti-symmetric.

Let $x, y, z \in X$. Assume that $x \preccurlyeq y$ and $y \preccurlyeq z$. Then, by definition, $f(x) \leq f(y)$ and $f(y) \leq f(z)$. These imply that $f(x) \leq f(z)$ and so $x \preccurlyeq z$. Therefore, $\preccurlyeq$ is transitive.

(6 pts) 2. Using **induction**, prove that $\left(1+\frac{1}{n}\right)^n < n$ for all $n \geq 3$.

Base case: We have that $\left(1+\frac{1}{3}\right)^3 = \frac{4^3}{3^3} = \frac{64}{27} < \frac{81}{27} = 3$ and hence the claim holds for $n = 3$.

Inductive step: Let $n \geq 3$ be a natural number. Assume that $\left(1+\frac{1}{n}\right)^n < n$. This assumption and the fact that $\frac{1}{n+1} < \frac{1}{n}$ together imply that

$$\left(1+\frac{1}{n+1}\right)^{n+1} = \left(1+\frac{1}{n+1}\right)^n \cdot \left(1+\frac{1}{n+1}\right) < \left(1+\frac{1}{n}\right)^n \cdot \left(1+\frac{1}{n}\right) < n \cdot \frac{n+1}{n} = n+1$$

Therefore, if $\left(1+\frac{1}{n}\right)^n < n$, then $\left(1+\frac{1}{n+1}\right)^{n+1} < n+1$.

By the principle of induction, we have that $\left(1+\frac{1}{n}\right)^n < n$ for all natural numbers $n \geq 3$.

(2+2 pts) 3. Suppose that E is a relation on $\mathbb{Z}$ which is **both** an equivalence relation and a partial ordering.

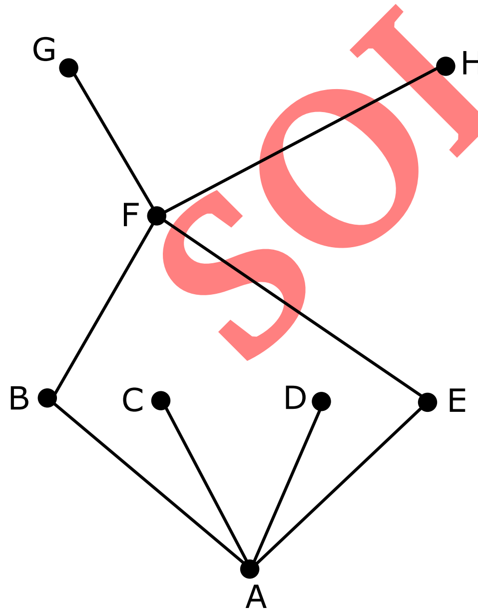
a) Show that if aEb then $a = b$.

Assume that aEb. Since E is an equivalence relation, it is symmetric and hence bEa. On the other hand, since E is a partial ordering, it is anti-symmetric and hence aEb and bEa together imply that $a = b$.

b) Show that E is not a total order relation. (**Hint.** You can use part a).)

It follows from the previous part of the question that if $a \neq b$, then $a \not E b$. In particular, we have that, for example, $111 \not E 112$ and $112 \not E 111$. If E were a total order, then we would have aEb or bEa for all $a, b \in \mathbb{Z}$. Hence, E cannot be a total order relation.

(2+2+2+2 pts) 4. Consider the partial ordering on the set $X = \{A, B, C, D, E, F, G, H\}$ whose Hasse diagram is given below. If they exist, find the following elements of X. (For this question only, you do **not** need to justify your answer.)



a) Maximal element(s)

G,H,C,D

b) Greatest element

There is no greatest element

c) Least upper bound of the subset $\{B, E\}$

F

d) Greatest lower bound of the subset $\{G, F, C\}$

A