M E T U Department of Mathematics

Math 111 Fundamentals of Mathematics Fall 2018 Midterm III 6.12.2018 17:40		
Last Name : Name : Student No :	Section : Duration : 65 *minutes*	
4 QUESTIONS ON 2 PAGES		TOTAL 30 POINTS

1	2	3	4			

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

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SOLUTION KEY

***(4+4 pts)* 1.**

a) Find a left inverse for the function $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(m) = m$ for all $m \in \mathbb{N}$.

Consider the function $g : \mathbb{Z} \to \mathbb{N}$ defined by

$$g(k) = \begin{cases} k & \text{if } k \in \mathbb{N} \\ 0 & \text{if } k \in \mathbb{Z} - \mathbb{N} \end{cases}$$

Then, for any $k \in \mathbb{N}$, we have that

$$(g \circ f)(k) = g(f(k)) = g(k) = k$$

Hence, $g \circ f = 1_{\mathbb{N}}$ and so g is a left inverse for f.

b) Let $f : A \to B$ be a function and let $Y \subseteq B$. Prove that if f is onto, then $Y \subseteq f(f^{-1}(Y))$.

Assume that f is onto. We wish to show that $Y \subseteq f(f^{-1}(Y))$. Let $b \in Y$. Since f is onto, there exists $a \in A$ such that $f(a) = b$. Then, by definition, as $f(a) \in Y$, we have that $a \in f^{-1}(Y)$.

By the definition of the image of a set under a function, we have that

$$f(f^{-1}(Y)) = \{y \in Y \mid \exists x \in f^{-1}(Y) \ \ f(x) = y\}$$

It then follows from $b = f(a)$ and $a \in f^{-1}(Y)$ that $b \in f(f^{-1}(Y))$. Therefore, $Y \subseteq f(f^{-1}(Y))$.

***(2+4 pts)* 2.**

a) Consider the set $A = \{1, 2, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$. Find the following set:

$A - \mathcal{P}(A) = \{1, 2, \{1, 2, 3\}\}$

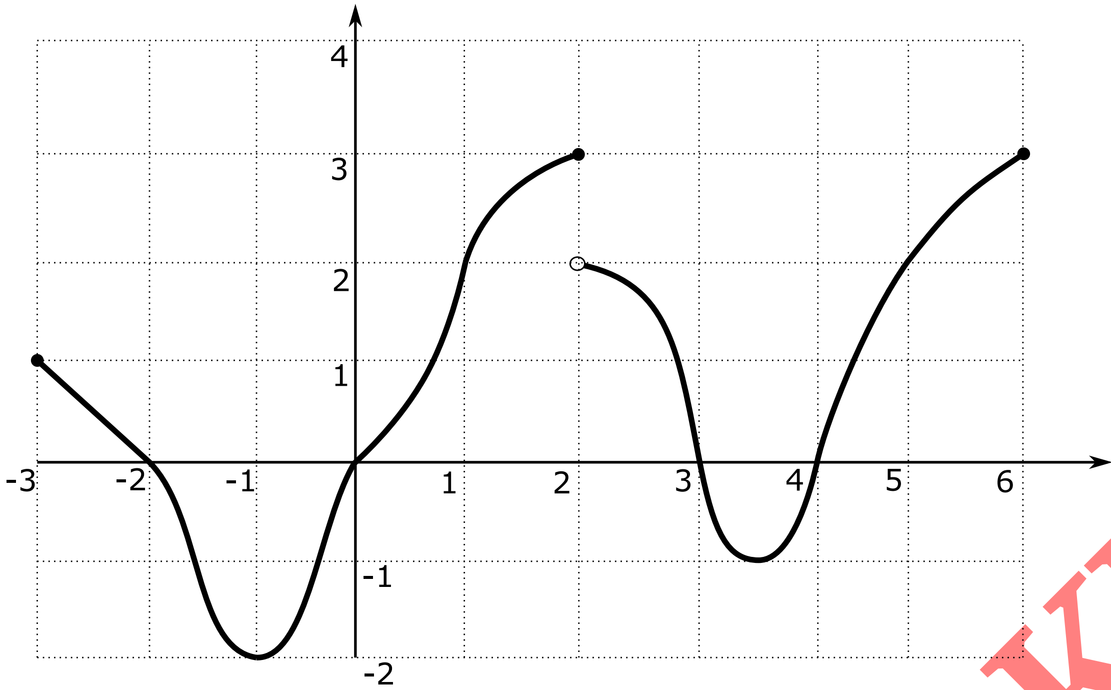
b) Show that the following statement is false:

For every set A, B, C, D, $\quad (A \times B) - (C \times D) = (A - C) \times (B - D)$.

We wish to prove that there exist sets A, B, C, D such that $(A \times B) - (C \times D) \neq (A - C) \times (B - D)$.

Choose $A = B = D = \{0\}$ and $C = \emptyset$. Then we have that $A \times B = \{(0, 0)\}$ and $C \times D = \emptyset$ and so $(A \times B) - (C \times D) = \{(0, 0)\}$. However, $A - C = \{0\}$ and $B - D = \emptyset$ and hence, $(A - C) \times (B - D) = \emptyset$. Therefore, $(A \times B) - (C \times D) \neq (A - C) \times (B - D)$.

***(2+2+2+2 pts)* 3.** Let $f : [-3, 6] \to \mathbb{R}$ be the function whose graph is given below:



Find the following sets. (For this question only, you do **not** need to prove your claim.)

a) $f\Big([-2,1]\Big) = [-2,2]$

b) $f^{-1}\Big((2,3)\Big) = (1,2) \cup (5,6)$

c) $f^{-1}\Big(\{4\}\Big) = \emptyset$

d) $f^{-1}\Big(\{0\}\Big) = \{-2,0,3,4\}$

***(3+3+2 pts)* 4.** Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined as follows:

$$f(x) = \begin{cases} x & \text{if } x \text{ is odd} \\ x+2 & \text{if } x \text{ is even} \end{cases}, \text{ for any } x \in \mathbb{Z}.$$

a) Show that f is one-to-one. Let $x, y \in \mathbb{Z}$. Assume that $f(x) = f(y)$. We wish to show that $x = y$. We first make the following observation. For all $i \in \mathbb{Z}$, we have that i is even if and only if $f(i)$ is even. We now split into two cases by considering the number $m = f(x) = f(y)$:

Case 1. If m is odd, then, by the observation, x and y are both odd. But then, $x = f(x) = f(y) = y$.

Case 2. If m is even, then, by the observation, x and y are both even. But then, $x + 2 = f(x) = f(y) = y + 2$ implies that $x = y$.

In both cases, we have that $x = y$. Thus, f is one-to-one.

b) Show that f is onto. Let $y \in \mathbb{Z}$. We split into two cases:

Case 1. If y is odd, then choose $x = y$. In this case, $x \in \mathbb{Z}$ and x is odd. Thus, $f(x) = x = y$.

Case 2. If y is even, then choose $x = y-2$. In this case, $x \in \mathbb{Z}$ and x is even. Thus, $f(x) = x+2 = (y-2)+2 = y$. In both cases, we have found $x \in \mathbb{Z}$ such that $f(x) = y$. Thus, f is onto.

c) Find f^{-1}. Since f is a bijection, it has an inverse f^{-1} which is equal to its right and left inverse. Let the function $g : \mathbb{Z} \to \mathbb{Z}$ be given by $g(x) = \begin{cases} x & \text{if } x \text{ is odd} \\ x-2 & \text{if } x \text{ is even} \end{cases}$, for any $x \in \mathbb{Z}$. Then, $(g \circ f)(x) = g(f(x)) = g(x) = x$ for all odd $x \in \mathbb{Z}$ and $(g \circ f)(x) = g(f(x)) = g(x+2) = (x+2)-2 = x$ for all even $x \in \mathbb{Z}$. Thus, $g \circ f = 1_{\mathbb{Z}}$ and so g is the left inverse of f, which implies that $g = f^{-1}$.