M E T U   Department of Mathematics

Math 111 Fundamentals of Mathematics Fall 2018 Midterm II 15.11.2018 17:40		
Last Name: : Name: : Student No: :	Section : Duration : 65 *minutes*	
4 QUESTIONS ON 2 PAGES		TOTAL 30 POINTS

1	2	3	4			

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(3+3 pts) **1.** Consider the set $A = \{1, 2, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$. Find the following sets:

- $A - \mathcal{P}(A) = \{1, 2, \{1, 2, 3\}\}$

- $A \cup \mathcal{P}(\{1, 2\}) = \{\emptyset, 1, 2, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$

(10 pts) **2.** Let A, B, C be sets. Prove that

$$(A \cap B) \subseteq C \quad \text{if and only if} \quad (A - C) \cap (B - C) = \emptyset$$

(**WARNING:** Drawing Venn diagrams is not a proof!)

- ($\Rightarrow$) We shall prove this direction a proof by contrapositive.

  Suppose that $(A-C)\cap(B-C) \neq \emptyset$. This means that there exists an element $a \in (A-C)\cap(B-C)$. So, $a \in A - C$ and $a \in B - C$. This implies that $a \in A$ and $a \in B$, but $a \notin C$. Hence, $a \in A \cap B$ but $a \notin C$. Therefore $(A \cap B) \not\subseteq C$.

- ($\Leftarrow$) We shall also prove this direction a proof by contrapositive.

  Suppose that $(A \cap B) \not\subseteq C$. This means that there exists an element $a \in A \cap B$ such that $a \notin C$. So, $a \in A$ and $a \in B$, but $a \notin C$. It then follows from the definition of set difference that $a \in A - C$ and $a \in B - C$. Consequently, we have that $a \in (A-C)\cap(B-C)$ and hence $(A-C)\cap(B-C) \neq \emptyset$.

***(3+4 pts)* 3.** Consider the statement:

For any integer m and for any integer n, if m is even and n is odd, then $mn + n$ is odd.

The following is an **incorrect** proof of this statement:

Proof: Let m be an integer and let $n = m+1$. Assume that m is even and n is odd. By assumption, since n is odd, $n = 2k+1$ for some integer k. It follows that $mn+n = (m+1) \cdot n = n \cdot n = (2k+1)(2k+1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Hence $mn + n$ is odd. $\square$

a) Find the mistake in this incorrect proof and explain why it is incorrect.

The statement asserts that the conclusion is true for **any** integers m and n. In the proof, m is arbitrarily chosen arbitrarily but the integer n is chosen as the successor of m and hence not arbitrary. Consequently, this proof shows that the conclusion is true for any two consecutive integers, and not for arbitrary integers.

b) Write a correct proof of this statement.

Let m and n be integers. Assume that m is even and n is odd. Then, by definition, there exist integers k, ℓ such that $m = 2k$ and $n = 2\ell + 1$. Hence, $mn + n = (2k)(2\ell + 1) + 2\ell + 1 = 2(2k\ell + k + \ell) + 1$. It follows that $mn + n$ is odd.

***(3+4 pts)* 4.** Let x, y be integers.

a) Prove the following statement by a **direct proof**: If x is even or y is even, then 4 divides $x^2 \cdot y^2$.

Suppose that x or y is even. We split into two cases.

Case 1, x is even: This means $x = 2k$ for some integer k. So, $x^2 \cdot y^2 = (2k)^2 \cdot y^2 = 4(k^2 y^2)$. Hence, 4 divides $x^2 \cdot y^2$.

Case 2, y is even: This means $y = 2m$ for some integer m. So, $x^2 \cdot y^2 = x^2 \cdot (2m)^2 = 4(x^2 m^2)$. Hence, 4 divides $x^2 \cdot y^2$.

In both cases, we have that 4 divides $x^2 \cdot y^2$.

b) Prove the following statement by a **proof by contrapositive**: If 4 divides $x^2 \cdot y^2$, then x is even or y is even.

We shall prove the contrapositive of the statement, that is, if x is odd and y is odd, then 4 does not divide $x^2 \cdot y^2$.

Suppose that x and y are odd integers. Then, by definition, $x = 2i + 1$ for some integer i and $y = 2j + 1$ for some integer j. It follows that $x^2 \cdot y^2 = (2i+1)^2(2j+1)^2 = (4i^2 + 4i + 1)(4j^2 + 4j + 1) = 4(4(i^2 + i)(j^2 + j) + (i^2 + i) + (j^2 + j)) + 1$ and so $x^2 \cdot y^2 = 4p + 1$ for some integer p. If it were the case that 4 divides $x^2 \cdot y^2$, then we would have that $x^2 \cdot y^2 = 4q$ for some integer q and so we would have $4(q - p) = 1$ for some integers p and q, which is a contradiction, since 4 does not divide 1. Hence, 4 does not divide $x^2 \cdot y^2$.