M E T U Department of Mathematics

Math 111 Fundamentals of Mathematics Fall 2018 Midterm I 25.10.2018 17:40		
Last Name : A hardworking Name : Student No : student	Signature : Section : Duration : 65 *minutes*	
4 QUESTIONS ON 2 PAGES		TOTAL 30 POINTS
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By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

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(2+2+2+2 pts) **1.** In this question, suppose that the possible values of the variables x and y are positive integers. Determine whether the following statements are true or false. In each case, briefly explain your reasoning.

a) $\forall x\, \forall y\; y \le x^2$ FALSE. For $x=1$ and $y=2$, the statement $2 \le 1^2$ is not true

b) $\forall x\, \exists y\; y \le x^2$ TRUE. For any positive integer x, choose $y=1$. Then we have that $1 \le x^2$.

c) $\exists x\, \forall y\; y \le x^2$ FALSE. We shall show that the negation of this statement $\forall x \exists y\; y > x^2$ is true. For any x, choose $y = x^2+1$. Then $y > x^2$ is true and so $\forall x \exists y\; y > x^2$ is true.

d) $\exists x\, \exists y\; y = x^2$ TRUE. Choose $x=1$ and $y=1$. Then we have that $y = x^2$

(3+3+3 pts) **2.** In this question, suppose that the possible values of all variables are positive integers. Moreover, suppose that $D(x,y)$ stands for "x divides y" and $P(x)$ stands for "x is prime".

a) Express the following statements using quantifiers, variables, logical connectives, positive integers, the symbols D and P.

- "Every positive integer which is prime divides some positive integer which is not prime."

$$\forall x\,\big(P(x) \to \exists y\,(D(x,y) \wedge \neg P(y))\big)$$

- "If there exists a positive integer which divides every positive integer, then every positive integer is prime; or some positive integer divides 2."

$$\Big[\big(\exists x\, \forall y\; D(x,y)\big) \to \forall z\, P(z)\Big] \vee \exists w\; D(w,2)$$

b) Find the negation of the following statement. $\exists x\Big[\forall y\Big(R(x,y) \to P(x)\Big) \wedge \exists z\Big(P(z) \wedge R(x,z)\Big)\Big]$

$$\forall x\Big[\exists y\,(R(x,y) \wedge \neg P(x)) \vee \forall z\,(\neg P(z) \vee \neg R(x,z))\Big]$$

(3+3 pts) **3.** Let P, Q and R be statements. Show that $P \to (Q \vee R) \iff (P \wedge \neg Q) \to R$

a) using a truth table.

P	Q	R	$(P \to (Q \vee R))$		$\longleftrightarrow$	$((P \wedge \neg Q) \to R)$	
T	T	T	T	T	T	F	T
T	T	F	T	T	T	F	T
T	F	T	T	T	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	T	F	T
F	T	F	T	T	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	F	T	F	T

Since the statement $(P \to (Q \vee R)) \leftrightarrow ((P \wedge \neg Q) \to R)$ is a tautology, we have that $(P \to (Q \vee R)) \iff (P \wedge \neg Q) \to R$

b) using the logical equivalences that we have covered in class.

$P \to (Q \vee R) \iff \neg P \vee (Q \vee R) \iff (\neg P \vee Q) \vee R \iff \neg (P \wedge \neg Q) \vee R \iff (P \wedge \neg Q) \to R$

(4+3 pts) **4.** a) Show that the following argument is **valid** by writing a **derivation**. At each step of your derivation, justify your reasoning by referring to relevant statements and inference rules.

$$\begin{array}{l} \neg P \wedge Q \\ (R \vee Q) \to (\neg S \to P) \\ W \to \neg S \\ \hline \neg W \end{array}$$

(1) $\neg P \wedge Q$

(2) $(R \vee Q) \to (\neg S \to P)$

(3) $W \to \neg S$

(4) Q from (1) and $\dfrac{A \wedge B}{B}$

(5) $R \vee Q$ from (4) and $\dfrac{B}{A \vee B}$

(6) $\neg S \to P$ from (2), (5) and modus ponens

(7) $W \to P$ from (3), (6) and $\dfrac{A \to B \quad B \to C}{A \to C}$

(8) $\neg P \to \neg W$ from (7) and contrapositive

(9) $\neg P$ from (1) and $\dfrac{A \wedge B}{A}$

(10) $\neg W$ from (8), (9) and modus ponens

b) Show that the following argument is **invalid**.

$$\begin{array}{l} \neg P \wedge Q \\ (R \vee Q) \to (\neg S \to P) \\ W \to \neg S \\ \hline W \end{array}$$

Let P be false, Q be true, R be true, S be false and W be false statements. Then $\neg P \wedge Q$, $(R \vee Q) \to (\neg S \to P)$ and $W \to \neg S$ are true statements but the conclusion W is false.