Department of Mathematics, METU, MATH 111 - FUNDAMENTALS OF MATHEMATICS

SECOND MIDTERM

STUDENT NUMBER	NAME, FAMILY NAME	GRADE

18th December 2015. Duration : 100 minutes.
Four questions : 13 + 12 , 8 + 8 + 9 , 12 + 8 + 5 , 10 + 5 + 5 + 5

1. Let A, B be sets and $S \subseteq A$ be nonempty.
(A) If $f : A \longrightarrow B$ is an injective function, prove that $f|_S : S \longrightarrow B$ is injective, too.
(B) Suppose that $g : A \longrightarrow B$ is a surjective function. Is $g|_S : S \longrightarrow B$ surjective ? Give a proof or a counterexample.

(A) Given $x, y \in S$, if $f|_S(x) = f|_S(y)$, then

$$f(x) = f|_S(x) = f|_S(y) = f(y)$$

hence $x = y$ since f is injective. This being true for arbitrary, $x, y \in S$, it follows that $f|_S$ is injective.

(B) For a surjective function $g : A \longrightarrow B$ and $S \subseteq A$, the function $g|_S$ is not surjective in general. For instance defining $g : \mathbb{R} \longrightarrow \mathbb{R}$ by $g(x) = x$ for each $x \in \mathbb{R}$ and taking $S = [0,1] \subseteq \mathbb{R}$, it is seen that $g|_S : S = [0,1] \longrightarrow \mathbb{R}$ is not surjective. Indeed $g|_S(S) = S = [0,1] \neq \mathbb{R}$.

2. Consider a function $f : A \longrightarrow B$ and subsets $M \subseteq A$, $N \subseteq B$. Prove the following :

(A) $M \subseteq f^{-1}\Big(f(M)\Big)$.

(B) $f\Big(f^{-1}(N)\Big) \subseteq N$.

(C) $f\Big(f^{-1}\Big(f(M)\Big)\Big) = f(M)$.

(A) If $x \in M$, then $f(x) \in f(M)$. Hence $x \in f^{-1}(f(M))$. This being true for arbitrary $x \in M$, it follows that $M \subseteq f^{-1}(f(M))$.

(B) If $y \in f(f^{-1}(N))$, then there exists $a \in f^{-1}(N)$ such that $y = f(a)$. As $a \in f^{-1}(N)$, $y = f(a) \in N$. This being true for arbitrary $y \in f(f^{-1}(N))$ it follows that

$$f^{-1}(f(N)) \subseteq N .$$

(C) Put $N' = f(M)$ and note that

$$(*) \qquad f(f^{-1}(f(M))) = f(f^{-1}(N')) \subseteq N' = f(M).$$

On the other hand, as $M \subseteq f^{-1}(f(M))$ by (A) it follows that

$$(**) \qquad f(M) \subseteq f(f^{-1}(f(M))) .$$

Combining $(*)$ and $(**)$

$$f(f^{-1}(f(M))) = f(M) .$$

3. Consider the relation $\mathfrak{R}$ on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ defined by

$$(x,y)\,\mathfrak{R}\,(x',y') \quad \text{if} \quad x' - x = 2(y' - y)\,.$$

for each $(x,y), (x',y') \in \mathbb{R}^2$.

(A) Verify that $\mathfrak{R}$ is an equivalence relation on $\mathbb{R}^2$.
(B) Describe the equivalence classes of $\mathfrak{R}$ geometrically.
(C) Do $(1,1)$ and $(-5,2)$ belong to the same $\mathfrak{R}$-equivalence class ?

(A) For arbitrary $(x,y) \in \mathbb{R}^2$, $0 = x - x = 2(y-y) = 0$.
Hence $(x,y)\,\mathcal{R}\,(x,y)$: Thus $\mathcal{R}$ is reflexive.
For arbitrary $(x,y), (x',y') \in \mathbb{R}^2$, if $(x,y)\,\mathcal{R}\,(x',y')$, hence $x' - x = 2(y'-y)$ and (multiplying both sides by -1 !) $x - x' = 2(y - y')$.
Thererifore $(x',y')\,\mathcal{R}\,(x,y)$: $\mathcal{R}$ is symmetric.
Given arbitrary $(x,y), (x',y'), (x'',y'') \in \mathbb{R}^2$ such that $(x,y)\,\mathcal{R}\,(x',y')$ and $(x',y')\,\mathcal{R}\,(x'',y'')$ equivalently $x' - x = 2(y'-y)$ and $x'' - x' = 2(y''-y')$ which give (upon addition !) $x'' - x = 2(y''-y)$, that is $(x,y)\,\mathcal{R}\,(x'',y'')$: $\mathcal{R}$ is transitive.

Consequently $\mathcal{R}$ is an equivalence relation !

(B) The equivalence class containing $(a,b) \in \mathbb{R}$ is

$$\{ \underset{}{(x,y)}^{\in \mathbb{R}^2} \mid x - a = 2(y - b) \}$$

or the "line" $2y - x + C = 0$ where $C = a - 2b$.
Indeed each equivalence class of $\mathcal{R}$ is a line of the form

$$2y - x + C = 0.$$

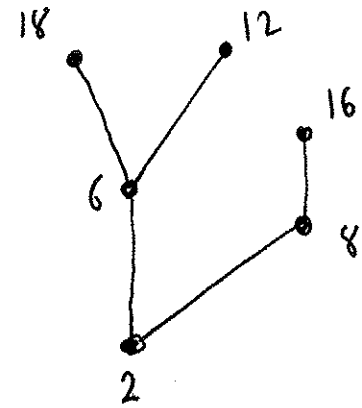
(C) $\underbrace{(1) - (-5)}_{= 6} \neq 2\underbrace{(1-2)}_{= -2}$

No ! $(1,1)$ and $(-5,2)$ do not belong to the same $\mathcal{R}$-equivalence class !

4. Let $A = \{2, 6, 8, 12, 16, 18\} \subseteq \mathbb{N}$ and consider the subset $W = \{6, 8\} \subseteq A$. Let $\preceq$ be the partial ordering defined by order relation of divisibility on A, to be precise for any $m, n \in A$, $m \preceq n$ if m divides n.

(A) Construct the Hasse diagram of the poset $(A, \preceq)$.
(B) What are the maximal and minimal elements in the poset $(A, \preceq)$?
(C) What are the greatest and least elements in the poset $(A, \preceq)$?
(D) What are the least upper and greatest lower bounds of W in the poset $(A, \preceq)$?

(A)



(B) 18, 12, 16 are maximal as they divide nothing except themselves.
2 is minimal as it is divisible by nothing except itself.

(C) There is no greatest element. 2 is the least element as it divides everything.

(D) The set of upper bounds of $\{6,8\}$ "W" is the empty set
This set has no least element.
Therefore W has no least upper bound!

The set of lower bounds of $W = \{6,8\}$ is $\{2\}$.
2 is the least element of this set.
Therefore 2 is the greatest upper bound of W.