Department of Mathematics, METU, MATH 111 - FUNDAMENTALS OF MATHEMATICS

FINAL EXAMINATION

STUDENT NUMBER	NAME, FAMILY NAME	GRADE

19th January 2016. Duration : 125 minutes.

Five questions : 15 + 5 , 7 + ~~8~~ + 7 , 20 , 20 , 4 + 4 + 4 + 4 + 4

6

1. Construct the truth table of the following statement :

$$\Big(P \longrightarrow (Q \longrightarrow R)\Big) \longrightarrow \Big(Q \longrightarrow (P \longrightarrow R)\Big)$$

Is this statement a tautology ? A contradiction ?

P	Q	R	$(P$	$\rightarrow$	$(Q$	$\rightarrow$	$R))$	$\longrightarrow$	$(Q$	$\rightarrow$	$(P$	$\rightarrow$	$R))$
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F	T	T	F	T	F	F
T	F	T	T	T	F	T	T	T	F	T	T	T	T
T	F	F	T	T	F	T	F	T	F	T	T	F	F
F	T	T	F	T	T	T	T	T	T	T	F	T	T
F	T	F	F	T	T	F	F	T	T	T	F	T	F
F	F	T	F	T	F	T	T	T	F	T	F	T	T
F	F	F	F	T	F	T	F	T	F	T	F	T	F

This statement is a <u>tautology</u>.

2. (A) Let $A = \{1,2,3,4,5,6,7\}$ and let $I_A : A \longrightarrow A$ be the identity function on A. Write down a function $f : A \longrightarrow A$ (explicitly !) such that $f \neq I_A$ and $f \circ f = I_A$.

(B) Does there exist a function $g : A \longrightarrow A$ such that $g \circ g = I_A$ and $g(x) \neq x$ for any $x \in A$?

(C) Let B be an arbitrary nonempty set and let $\mathfrak{P}(B)$ be the power set of B. Let $h : \mathfrak{P}(B) \longrightarrow \mathfrak{P}(B)$ be defined by $h(X) = B - X$ for any $X \in \mathfrak{P}(B)$. Show that h is bijective and find h^{-1}.

(A) $f(1) = 1$

$f(2) = 3$ & $f(3) = 2$

$f(4) = 5$ & $f(5) = 4$

$f(6) = 7$ & $f(7) = 6$

Clearly $f \neq I_A$ & $f \circ f = I_A$.

(B) Such a function is impossible: If such a $g : A \longrightarrow A$ existed then A would be the union of disjoint pairs of the form $\{x, g(x)\}$ and A would necessarily contain an <u>even</u> number of elements. 7 is odd.

(C) Clearly $h \circ h(X) = B - (B - X) = X$ for any $X \in \mathcal{P}(B)$ (or $X \subseteq B$!).

Hence $h \circ h = I_{\mathcal{P}(B)}$. ~~In other words~~

h has a left & right inverse and h is bijective. Obviously

$$h^{-1} = h.$$

3. Prove that

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

$x \in (A - B) \cup (B - A)$ iff $x \in A - B$ or ($\vee$) $x \in B - A$

iff $(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$

iff (by the distributivity of $\vee$ over $\wedge$)

$$(x \in A \vee x \in B) \wedge \underbrace{(x \in A \vee x \notin A)}_{\text{tautology}} \wedge \underbrace{(x \notin B \vee x \in B)}_{\text{tautology}} \wedge (x \notin B \vee x \notin A)$$

iff

$$(x \in A \vee x \in B) \wedge (x \notin B \vee x \notin A)$$

iff $(x \in A \cup B) \wedge \neg(x \in A \wedge x \in B)$

iff $(x \in A \cup B) \wedge (x \notin A \cap B)$

iff $x \in (A \cup B) - (A \cap B)$.

This being true for all $x \in (A - B) \cup (B - A)$....

4. For which values of $n \in \mathbb{N}$ is
$$3^n - 11 \leq n!$$
true ?

<u>True</u> for $n = 1$: $3^1 - 11 = -8 \leqslant 1! = 1$

" " $n = 2$: $3^2 - 11 = -2 \leqslant 2! = 2$

False " $n = 3$: $3^3 - 11 = 16 \leqslant 3! = 6$

" " $n = 4$: $3^4 - 11 = 70 \leqslant 4! = 24$

" " $n = 5$ $3^5 - 11 = 242 \leqslant 5! = 120$

True " $n = 6$ $3^6 - 11 = 718 \leqslant 6! = 720$

Moreover <u>if</u> $3^n - 11 \leqslant n!$ for some $n \geqslant 4$

<u>then</u>

$$\begin{aligned} 3^{n+1} - 11 = 3(3^n - 11) + 33 - 11 &= 3(3^n - 1) + 22 \\ &\leqslant 3 \cdot n! + 22 \\ &\leqslant 4\, n! \\ &\leqslant (n+1)! \end{aligned}$$

It follows that $3^n - 11 \leqslant n!$ holds <u>for all $n \in \mathbb{N}$</u>

<u>except</u> $n = 3, 4, 5$!

5. Indicate by writing T or F in the appropriate box whether you think the following assertions are true (T) or false (F). Do not try to justify !

(A) $\mathbb{R}$ is countable.	(F)
(B) $\mathbb{R} - \mathbb{Q}$ and $\mathbb{R}$ have the same cardinality.	(T)
(C) $\mathfrak{P}(\mathbb{N})$ is not countable.	(T)
(D) If $A \sim A'$ and $B \sim B'$, then $A \cup B \sim A' \cup B'$.	(F)
(E) $\mathbb{Z} \times \mathbb{Z}$ has the same cardinality as $\mathbb{Q}$.	(T)