2013 / MT1

M E T U
Department of Mathematics

Group	Fundamentals of Mathematics Midterm 1	List No.

Code : *Math 111* Acad. Year : *2013* Semester : *Fall* Instructor : *G.Ercan, S.Finashin, M.Kuzucuoğlu, Ö.Küçüksakallı.* Date : *November 7, 2013* Time : *17:40* Duration : *100 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 6 PAGES 60 TOTAL POINTS

1	2	3	4	5	6		

1. (10pts) Using truth tables, determine if the statement $(P \to \neg Q) \leftrightarrow (R \to (\neg Q \vee P))$ is a tautology, a contradiction, or neither.

Solution

P	Q	R	$(P$	$\to$	$\neg Q)$	$\leftrightarrow$	$(R$	$\to$	$(\neg Q$	$\vee$	$P))$
T	T	T	T	F	F	$\underline{F}$	T	T	F	T	T
T	T	F	T	F	F	$\underline{F}$	F	T	F	T	T
T	F	T	T	T	T	$\underline{T}$	T	T	T	T	T
T	F	F	T	T	T	$\underline{T}$	F	T	T	T	T
F	T	T	F	T	F	$\underline{F}$	T	F	F	F	F
F	T	F	F	T	F	$\underline{T}$	F	T	F	F	F
F	F	T	F	T	T	$\underline{T}$	T	T	T	T	F
F	F	F	F	T	T	$\underline{T}$	F	T	T	T	F

Since the formula takes both values "T" and "F", the statement is neither a tautology nor a contradiction.

2. (10pts) Assume that x and y take real values. Determine whether the following statements (1)–(4) are true or false. Explain your answers briefly.

$$P(x, y) = \text{“}x^2 + y = 5\text{''}$$

1. $\exists x\ \forall y\ P(x, y)$

   Solution: "There exists x such that for all y we have $x^2 + y = 5$."

   This is false, because for any given value of x the identity $x^2 + y = 5$ is true just for one (and not all) values of y.

2. $\forall y\ \exists x\ P(x, y)$

   Solution: "For all y there exists x such that $x^2 + y = 5$."

   This is false, because if we take $y > 5$ then $x^2 = 5 - y < 0$, and we cannot find any x satisfying the identity.

3. $\exists y\ \forall x\ P(x, y)$

   Solution: "There exists y such that for all x we have $x^2 + y = 5$."

   This is false, because for any given value of y the identity $x^2 + y = 5$ is true not more than for two (and not all) values of x.

4. $\forall x\ \exists y\ P(x, y)$

   Solution: "For all x there exists y such that $x^2 + y = 5$."

   This is true, because for any given value of x we choose $y = 5 - x^2$, which is a real number, and the required identity is satisfied.

3. (10pts) Give a derivation for the following argument (which is known to be valid).

$$\begin{array}{l} P \to (Q \to R) \\ T \to Q \\ P \vee S \\ \underline{\neg S \qquad\qquad} \\ \neg R \to \neg T \end{array}$$

Solution

(1) $P \to (Q \to R)$		
(2) $T \to Q$		
(3) $P \vee S$		
(4) $\underline{\neg S}$		
(5) P	$(3), (4)$,	*Modus Tollendo Ponens*
(6) $Q \to R$	$(1), (6)$,	*Modus Ponens*
(7) $T \to R$	$(2), (6)$,	*Hypothetical Syllogism*
(8) $\neg R \to \neg T$	(7),	*Contrapositive*

4. (10pts) Let a, b and c be integers. Prove that if a does not divide bc, then a does not divide b.

Solution Proof is by contrapositive. Namely we show that if a divides b, then a divides bc. So assume that a divides b. Then there exists $k \in \mathbb{Z}$, such that $b = ak$. Then multiplying both sides of the equality by c we obtain

$$\begin{aligned} bc &= (ak)c \\ &= a(kc) \quad \text{by associativity in } \mathbb{Z} \end{aligned}$$

Since $kc \in \mathbb{Z}$. This shows that a divides bc. Hence we are done.

5. (10pts) Prove that the following three statements about an integer n are equivalent.

1. $3 \nmid n$
2. $3 \mid n^2 - 1$
3. there exists an integer k such that $n = 3k + 1$, or $n = 3k + 2$.

Solution We will show that these statements are equivalent by showing $1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$.

($1 \Rightarrow 3$) Suppose that n is not divisible by 3. Then there are two possibilities. Either $n = 3k + 1$ or $n = 3k + 2$ for some integer k.

($3 \Rightarrow 2$) If $n = 3k+1$, then $n^2 - 1 = 9k^2 + 6k$ and it is divisible by 3. If $n = 3k+2$, then $n^2 - 1 = 9k^2 + 12k$ and it is divisible by 3 as well. We conclude that $n^2 - 1$ is divisible by 3 in either case.

($2 \Rightarrow 1$) We prove this part by contrapositive. Suppose that 3 divides n. Then there exists an integer k such that $n = 3k$. Thus $n^2 - 1 = 9k^2 - 1$ and it is not divisible by 3.

6. **(10pts)** Let A, B and C be sets. Prove that if $A \subseteq B$ and $B \cap C = \varnothing$, then $A \cap C = \varnothing$.

Solution (Proof by contradiction) Assume that $A \cap C$ is nonempty. Then there is an element $x \in A \cap C$. It follows that $x \in A$ and $x \in C$. Since $A \subseteq B$, we must have $x \in B$. Now $x \in B$ and $x \in C$. As a result $x \in B \cap C$. However this is contradiction to the hypothesis $B \cap C = \varnothing$.

Solution (Direct Proof) If A is the empty set then the conclusion $A \cap C = \varnothing$ is always true and there is nothing to prove. If A is not empty, then pick an arbitrary element $a \in A$. Since $A \subseteq B$, we must have $a \in B$. Using the hypothesis $B \cap C = \varnothing$, we find that $a \notin C$. Recall that a is an arbitrary element of A and it is not in C. Therefore we can conclude that $A \cap C = \varnothing$.