2012 MT2

METU KEY

Department of Mathematics

Group	Fundamentals of Mathematics Midterm 2	List No.

Code : *Math 111* Acad. Year : *2012* Semester : *Fall* Instructor : *Finashin, Solak, Kuzucuoğlu, Küçüksakallı.* Date : *December 6, 2012* Time : *17:40* Duration : *100 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 4 PAGES 60 TOTAL POINTS

1	2	3	4	5	6		

1. (10pts) Let A and B be sets and let X be a subset of B. Given a function $f : A \to B$,

(a) prove that $f(f^{-1}(X)) \subseteq X$,

Let $y \in f(f^{-1}(X))$. ~~becau~~ Then $\exists t \in f^{-1}(X)$ such that $f(t) = y$. Since $t \in f^{-1}(X)$, $\underset{y}{\underset{\parallel}{f(t)}} \in X$.

Thus, $y \in X$ and $f(f^{-1}(X)) \subseteq X$.

(b) prove that $f(f^{-1}(X)) = X$ if f is surjective.

By part(a) $f(f^{-1}(X)) \subseteq X$. We only need to show that $X \subseteq f(f^{-1}(X))$.

Let $x \in X$ be any element. Since f is surjective, $\exists a \in A$ such that $f(a) = x$. This means that, $a \in f^{-1}(X)$. Hence $\underset{x}{\underset{\parallel}{f(a)}} \in f(f^{-1}(X))$.

$\Rightarrow X \subseteq f(f^{-1}(X))$.

2. (10pts) Let $f : A \to B$ be a function and let A_1, A_2 be subsets of A.

(a) Show that $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ when f is injective.

"$\Rightarrow$" Let $y \in f(A_1 \cap A_2)$. Then $\exists x \in (A_1 \cap A_2)$ such that $f(x) = y$.

$x \in A_1 \cap A_2 \Rightarrow x \in A_1$ and $x \in A_2$. Then $f(x) = y \in f(A_1)$

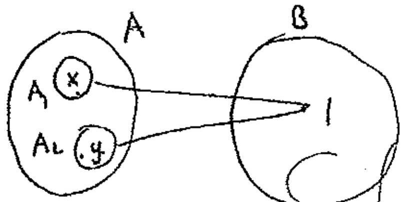
and $f(x) = y \in f(A_2)$. Hence $y \in f(A_1) \cap f(A_2)$.

"$\Leftarrow$" Let $y \in f(A_1) \cap f(A_2)$. Then $y \in f(A_1)$ and $y \in f(A_2)$.

$\exists x_1 \in A_1$ such that $f(x_1) = y$ and $\exists x_2 \in A_2$ st. $f(x_2) = y$

Since f is injective $x_1 = x_2$ and $x_1 \in A_1 \cap A_2$, this implies that $f(x_1) = y \in f(A_1 \cap A_2)$.

(b) If f is not injective then show that the equality $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ may not hold.



f is not 1-1.

$A_1 \cap A_2 = \phi$

Solution of (6 and 3)

$f(\phi) = \phi$ and $f(A_1) \cap f(A_2) = \{1\}$

Thus, $f(A_1 \cap A_2) = f(\phi) = \phi \neq f(A_1) \cap f(A_2) = \{1\}$.

3. (10pts) Let $\sim$ be an equivalence relation on a non-empty set A and let $[\alpha]$ denote the equivalence class of α. If $[\alpha] \cap [\beta] \neq \emptyset$, then show that $[\alpha] = [\beta]$ by using the definitions only.

Let $x \in [\alpha] \cap [\beta]$. Then $\alpha \sim x$ and $\beta \sim x$. By symmetry $\alpha \sim x$ and $x \sim \beta$. Transitivity of $\sim$ implies that $\alpha \sim \beta$. Hence $\alpha \in [\beta]$. If $t \in [\alpha]$, then $t \sim \alpha$. Then $t \sim \alpha$ and $\alpha \sim \beta$ implies $t \sim \beta$. Hence $t \in [\beta]$.

It follows that $[\alpha] \subseteq [\beta]$ $(*)$

If $y \in [\beta]$, then $y \sim \beta$. Then $y \sim \beta$ and $\beta \sim \alpha$ implies $y \sim \alpha$. Hence $y \in [\alpha]$. It follows that

$[\beta] \subseteq [\alpha]$. $(**)$

From $(*)$ and $(**)$ we obtain $[\alpha] = [\beta]$

4. (10pts) Let $\sim$ be a relation on $\mathbb{R} \times \mathbb{R}$ given by $(a, b) \sim (c, d)$ if and only if $a + b = c + d$ for all $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$.

(a) Show that $\sim$ is an equivalence relation.

(1) $(a,b) \sim (a,b)$, since $a+b=a+b$ $\Rightarrow$ relation "$\sim$" is reflexive

(2) if $(a,b) \sim (c,d)$, then $a+b=c+d$, so, $c+d=a+b$ and $(c,d) \sim (a,b)$
so, relation "$\sim$" is symmetric

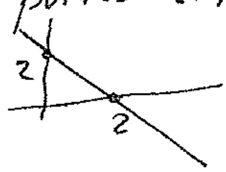
(3) if $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$ then $a+b=c+d$ and $c+d=e+f$,
thus, $a+b=e+f$ and $(a,b) \sim (e,f)$. So, relation "$\sim$" is transitive

Since "$\sim$" is reflexive, symmetric and transitive, it is an equivalence relation.

(b) Determine the equivalence class $[(1,1)]$ and interpret the geometrical meaning of the points in the class $[(1,1)]$.

$[(1,1)] = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x+y = 1+1 \}$

This class consists of the points $(x,y) \in \mathbb{R}^2$ lying on the line $x+y=2$



5. (10pts) Suppose that $B \subseteq A$. Prove that

$$(A \times A) - (B \times B) = [(A - B) \times B] \cup [A \times (A - B)].$$

"$\subseteq$" if $(x,y) \in A \times A - (B \times B)$, then $(x,y) \in A \times A$ and $(x,y) \notin B \times B$
that is, $x, y \in A$ and either $x \notin B$ or $y \notin B$

Case 1 $y \in B$, then $x \notin B$ and thus, $(x,y) \in (A-B) \times B$

Case 2 $y \notin B$, then $(x,y) \in A \times (A-B)$

Thus, $(x,y) \in [(A-B) \times B] \cup [A \times (A-B)]$

"$\supseteq$" if $(x,y) \in [(A-B) \times B] \cup [A \times (A-B)]$, then either

Case 1 $(x,y) \in (A-B) \times B$, that is $x \in A-B$ and $y \in B$, or

Case 2 $(x,y) \in A \times (A-B)$, that is $x \in A$ and $y \in A-B$

In the Case 1, $x \notin B$, so $(x,y) \notin B \times B$, but $(x,y) \in A \times A$

In the case 2, $y \notin B$, so $(x,y) \notin B \times B$, but $(x,y) \in A \times A$

So, in any case, $(x,y) \in (A \times A) - (B \times B)$

SOLUTION KEY (6)

6. (10pts) Two integers a and b are called *relatively prime* if their greatest common divisor is 1. Let

$$L = \{(a, b) \in \mathbb{N} \times \mathbb{N} : a \text{ and } b \text{ are relatively prime}\}.$$

Consider the function $f : L \to L$ defined by $f(a, b) = (a + b, b)$.

(5) (a) If a and b are relatively prime, then show that $a + b$ and b are relatively prime.

Assume that $a+b$ and b are not relatively prime. Then there exists a prime p such that $p \mid a+b$ and $p \mid b$. Then $p \mid (a+b)-b = a$. This is a contradiction as a and b are relatively prime.

(2) (b) Is f injective? Explain your answer.

Yes $f(a_1, b_1) = f(a_2, b_2)$ implies $(a_1+b_1, b_1) = (a_2+b_2, b_2)$

Then $b_1 = b_2$. This shows that $a_1+b_2 = a_2+b_2$ Hence

$a_1+b_1 = a_2+b_2$

$a_1 = a_2$

(2) (c) Is f surjective? Explain your answer.

NO Given (a, b) relatively prime. Then

$f(x, y) = (a, b)$ implies $(x+y, y) = (a, b)$. Then $y = b$

$x+y = a$. Hence $x = a-y = a-b$ But $(a-b, b)$ may not be in L as $a-b$ may not be in $\mathbb{N}$ as

$(2,5) \in L$ but $(2-5, 5) = (-3, 5) \notin L$

(1) (d) Is f bijective? Explain your answer.

No as f is not surjective.