2012 MT 1

METU
Department of Mathematics

Group	Fundamentals of Mathematics Midterm 1	List No.

Code : *Math 111* Acad. Year : *2012* Semester : *Fall* Instructor : *S.Finashin, E.Solak, M.Kuzucuoğlu, O.Küçüksakallı.* Date : *November 1, 2012* Time : *17:40* Duration : *90 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 4 PAGES 60 TOTAL POINTS

1	2	3	4	5	6		

1. **(10pts)** Determine if the following argument is valid or not. If it is valid give a derivation, if it is not, show why.

$$\begin{array}{l} (\neg P \vee \neg Q) \to R \\ R \to T \\ \underline{\neg T \qquad\qquad} \\ P \end{array}$$

Solution

(1)	$(\neg P \vee \neg Q) \to R$	
(2)	$R \to T$	
(3)	$\neg T$	
(4)	$\neg R$	(2), (3), *Modus Tollens*
(5)	$\neg(\neg P \vee \neg Q)$	(1), (4), *Modus Tollens*
(6)	$\neg\neg P \wedge \neg\neg Q$	(5), *De Morgan Law*
(7)	$P \wedge Q$	(6), *Double negation*
(8)	P	(7), *Simplification*

2. **(10pts)** Using truth tables, determine if the statement $P \leftrightarrow (P \vee (P \wedge \neg Q))$ is a tautology, a contradiction, or neither.

Solution

P	Q	P	$\leftrightarrow$	$(P$	$\vee$	$(P$	$\wedge$	$\neg Q))$
T	T	T	$\underline{T}$	T	T	T	F	F
T	F	T	$\underline{T}$	T	T	T	T	T
F	T	F	$\underline{T}$	F	F	F	F	T
F	F	F	$\underline{T}$	F	F	F	F	T

Since the statement takes only "True" value, it is a tautology.

3. **(10pts)** Simplify the negation $\neg[\forall x\ \exists y\ (P(x) \wedge Q(y)) \to \neg R(x,y)]$ by finding an equivalent statement that does not contain the negation symbol "$\neg$". Show the steps of your solution.

Solution

$$\neg[\forall x\ \exists y\ (P(x) \wedge Q(y)) \to \neg R(x,y)] \iff$$
$$\exists x\ \neg[\exists y\ (P(x) \wedge Q(y)) \to \neg R(x,y)] \iff$$
$$\exists x\ \forall y\ \neg[(P(x) \wedge Q(y)) \to \neg R(x,y)] \iff$$
$$\exists x\ \forall y\ \neg[\neg(P(x) \wedge Q(y)) \vee \neg R(x,y)] \iff$$
$$\exists x\ \forall y\ \neg\neg(P(x) \wedge Q(y)) \wedge \neg\neg R(x,y) \iff$$
$$\exists x\ \forall y\ (P(x) \wedge Q(y)) \wedge R(x,y)$$

4. (10pts) Let x and y be positive integers. Determine whether the following statements are true or false. Explain your answers briefly.

1. $\forall x \; \forall y \; (x < y)$

   Solution
   False. If $x = 2$ and $y = 1$, then $x < y$ is not true.

2. $\forall x \; \exists y \; (x < y)$

   Solution
   True. Given x, choose $y = x + 1$. Then $x < y$.

3. $\exists y \; \forall x \; (x < y)$

   Solution
   False. Suppose such y exists. Then for $x = y + 1$, the statement $x < y$ will be false. This is a contradiction.

4. $\exists x \; \exists y \; (x < y)$

   Solution
   True. Choose $x = 1$ and $y = 2$, then $x < y$ is true.

5. $\forall y \; \exists x \; (x < y)$

   Solution
   False. If $y = 1$ then there is no positive integer x such that $x < y$.

6. (10pts) Let a and b be integers. Prove that $a+b$ is even if and only if a^2+b^2 is even. (Pay attention to logical presentation of your solution.)

Solution

($\Rightarrow$): Suppose that $a+b$ is even. Then there exists an integer k such that $a+b=2k$. We have $a^2+2ab+b^2=4k^2$ and therefore $a^2+b^2=2(2k^2-ab)$. We conclude that a^2+b^2 is even.

($\Leftarrow$): We prove this part by contrapositive. Suppose that $a+b$ is odd. Then there exists an integer k such that $a+b=2k+1$. We have $a^2+2ab+b^2=4k^2+4k+1$ and therefore $a^2+b^2=2(2k^2+2k-ab)+1$. We conclude that a^2+b^2 is odd.

5. (10pts) Assume that a is an integer such that a^2-1 is not divisible by 3. Prove that a is divisible by 3.

Solution

Suppose that a is an integer such that a^2-1 is not divisible by 3. We use proof by contradiction. Assume that a is an integer not divisible by 3. There are two cases. Either $a=3k+1$ or $a=3k+2$ for some integer k. If $a=3k+1$, then $a^2-1=9k^2+6k+1-1=3(3k^2+2k)$ is divisible by 3, a contradiction. Similarly if $a=3k+2$, then $a^2-1=9k^2+12k+4-1=3(3k^2+4k+1)$ is divisible by 3, another contradiction. Therefore we conclude that if a^2-1 is not divisible by 3, then a must be divisible by 3.