M E T U
Department of Mathematics

Group	Fundamentals of Mathematics Final		List No.
Code : *Math 111* Acad. Year : *2012* Semester : *Fall* Instructor : *Finashin, Solak, Kuzucuoğlu, Küçüksakallı.* Date : *January 15, 2013* Time : *13:30* Duration : *120 minutes*	Last Name : Name : Department : Signature :	Student No. : Section :	
	6 QUESTIONS ON 4 PAGES 80 TOTAL POINTS		

1	2	3	4	5	6		

2012

1. (15pts) Show that the set of natural numbers $\mathbb{N}$ has the same cardinality as the set of integers $\mathbb{Z}$.

In order to show $\mathbb{N} \sim \mathbb{Z}$, we need to find a bijective function $f: \mathbb{N} \to \mathbb{Z}$.

Consider the function defined by

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -\frac{(n-1)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

<u>f is one-to-one:</u> Suppose that $f(n_1) = f(n_2)$

If $f(n_1) > 0$, then $\frac{n_1}{2} = \frac{n_2}{2}$. It follows that $n_1 = n_2$

If $f(n_1) \leq 0$, then $\frac{-(n_1-1)}{2} = -\frac{(n_2-1)}{2}$. Thus $n_1 = n_2$

We conclude that f is one-to-one.

<u>f is onto:</u> Let m be an arbitrary element of $\mathbb{Z}$.

If $m > 0$, then $2m \in \mathbb{N}$ and $f(2m) = m$

If $m \leq 0$, then $-2m+1 \in \mathbb{N}$ and $f(-2m+1) = m$

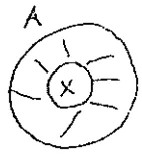
We conclude that f is onto.

2. **(10pts)** Let A be a non empty set and let $\mathcal{P}(A)$ denote the power set of A. Consider the function

$$\begin{array}{rccc} f: & \mathcal{P}(A) & \to & \mathcal{P}(A) \\ & X & \mapsto & A - X. \end{array}$$

(a) Is f one-to-one? Justify your answer.

Yes. f is one-to-one. Assume $f(X) = f(Y)$ for, $X, Y \in \mathcal{P}(A)$. Then $A - X = A - Y$. This implies $A - (A - X) = A - (A - Y)$



$\Rightarrow X = Y \Rightarrow f$ is 1-1.

(b) Is f onto? Justify your answer.

Yes. f is onto. Let Y be any element in $\mathcal{P}(A)$. Then $A - (A - Y) = Y$, i.e., $f(A - Y) = Y$.

3. **(15pts)** Consider the sequence $a_1, a_2, a_3, \ldots$ recursively defined by $a_{n+2} = 3a_{n+1} - 2a_n$ for all $n \in \mathbb{N}$ where $a_1 = 1$ and $a_2 = 3$.

(a) Find a_n for $n \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$a_1 = 1$, $a_2 = 3$, $a_3 = 7$, $a_4 = 15$, $a_5 = 31$, $a_6 = 63$, $a_7 = 127$,

$a_8 = 255$

(b) Find an explicit formula for the sequence a_n and prove that your formula satisfies the equation $a_{n+2} = 3a_{n+1} - 2a_n$ for all $n \in \mathbb{N}$.

$a_n = 2^n - 1$ is the explicit formula for a_n. Then

$$\begin{aligned} 3a_{n+1} - 2a_n &= 3(2^{n+1} - 1) - 2(2^n - 1) \\ &= 3.2^{n+1} - 3 - 2^{n+1} + 2 \\ &= 2.2^{n+1} - 1 = 2^{n+2} - 1 = a_{n+2} \end{aligned}$$

by the formula of a_n

Hence, $a_{n+2} = 3a_{n+1} - 2a_n$.

4. **(10pts)** Let $\mathbb{R}^* = \mathbb{R} - \{0\}$ be the set of all non-zero real numbers. Let T be a relation on $\mathbb{R}^*$ defined by $x\,T\,y$ if and only if $y = ax$ for some $a \in \mathbb{R}^*$.

(a) Is T an equivalence relation? Justify your answer.

(i) xTx as $x = x \cdot 1$ $1 \in \mathbb{R}^*$

(ii) If xTy, then $y = xa$ for some $a \in \mathbb{R}^*$. Then $x = ya^{-1}$ where $a^{-1} \in \mathbb{R}$ so yTx.

(iii) If xTy and yTz, then $y = xa$ and $z = yb$ where $a, b \in \mathbb{R}^*$. Then

$z = xab = x(ab)$ where $ab \in \mathbb{R}^*$. Hence xTz.

Since T is reflexive, symmetric and transitive, T is an equivalence relation.

(b) Is T a partial order? Justify your answer. T is not a partial order as

$2\,T\,3$ as $3 = 2 \cdot \frac{3}{2}$ and $3\,T\,2$ as $2 = 3 \cdot \frac{2}{3}$

but $2 \neq 3$.

5. **(15pts)** Prove that $n! > 2^n$ for all natural numbers $n \geq 4$.

Proof by induction on n. For $n = 4$

$4! = 24 > 2^4 = 16$. So the inequality is true for $n=4$

Assume that $n! > 2^n$ (*)

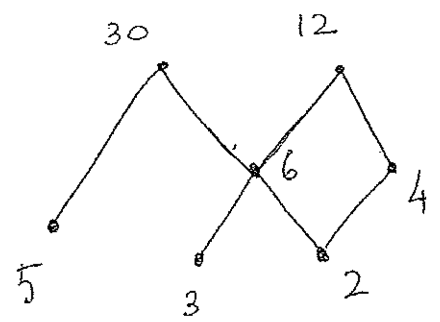
Then $(n+1)! = n!\,(n+1) > 2^n \cdot (n+1)$ by induction assumption (*)

But $n \geq 4$ implies $n+1 \geq 5$ Hence

$(n+1)! > 2^n(n+1) \geq 2^n \cdot 2 = 2^{n+1}$. Hence by PMI the inequality is true for all $n \geq 4$.

6. **(15pts)** Let $A := \{2,3,4,5,6,12,30\}$ be the partially ordered set with the divisibility relation for $a, b \in A$, $a|b$ equivalently $b = ak$ for some $k \in \mathbb{N}$.

(a) Draw a Hasse diagram for the poset A.



(b) Find all minimal and maximal elements if there are any.

Minimal elements are $\{2,3,5\}$. (elements which can not be divided by any other element of A)

Maximal element sare $\{12,30\}$. (elements which does not divide any other element of A)

(c) Is there a greatest lower bound and a least upper bound for A in $\mathbb{N}$ with usual divisibility $a|b$ in $\mathbb{N}$? If so what are they?

Greatest lower bound is 1. (biggest element in $\mathbb{N}$ dividing all elements in A)

Least upper bound is 60. (smallest element in $\mathbb{N}$ divisible by all elements in A)