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METU MATH 111, EXAM 2

Tuesday, December 13, 2010, at 17:40

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Instructions: There are 7 numbered problems on 4 pages. Please work carefully. It should be obvious to the grader how to read your solutions.

Problem 1. Write down a bijection from the interval $(1, 2)$ to $\mathbb{R}$. (You need not prove that it is a bijection.)

$(1,2) \to (-\frac{\pi}{2}, \frac{\pi}{2})$

$x \longmapsto \pi x - \frac{3\pi}{2}$

$(-\pi/2, \pi/2) \to \mathbb{R}$

$x \longmapsto \tan x$

$f(x) = \tan\left(\pi x - \frac{3\pi}{2}\right)$

Problem 2. In this problem,

- $\mathscr{P}(A)$ stands for the power set of A,
- S is the set of polynomials in the variable x with integer coefficients,
- $T = \{\pi^k + n \colon k, n \in \mathbb{N}\}$ (where π is the usual irrational constant).

Let

$$\Omega = \{\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{N} \times \mathbb{R}, \mathbb{N} \times \mathbb{Z}, \mathbb{R} \smallsetminus \mathbb{Q}, \mathscr{P}(\mathbb{R}), \mathscr{P}(\mathbb{Q}), S, T\}.$$

It is known that the partition of Ω with respect to equipollence ($\approx$) can be written as $\{A_0, A_1, A_2\}$. Find the sets A_0, A_1, A_2. (You are not required to prove your answer; but you will lose points if you puts elements of Ω into the wrong sets A_i.)

$A_0 = \{\mathbb{N}, \mathbb{Q}, \mathbb{N}\times\mathbb{Z}, S, T\}$

$A_1 = \{\mathbb{R}, \mathbb{N}\times\mathbb{R}, \mathbb{R}\setminus\mathbb{Q}, \mathcal{P}(\mathbb{Q})\}$

$A_2 = \{\mathcal{P}(\mathbb{R})\}$

Problem 3. Let A, B, C, and D be subsets of some universal set $\mathcal{U}$.

(a) Prove that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

If $(A\times B)\cup(C\times D)=\emptyset$, then we are done. Otherwise pick an arbitrary $(x,y) \in (A\times B)\cup(C\times D)$. Now $(x,y)\in A\times B$ or $(x,y)\in C\times D$. So $x\in A$ or $x\in C$. and $y\in B$ or $y\in D$. It follows that $x\in A\cup C$ and $y\in B\cup D$. Therefore $(x,y)\in (A\cup C)\times(B\cup D)$.

(b) Give an example where the inclusion in (a) is proper.

Pick $A=D=\emptyset$ and $B=C=\{1\}$. Then

$$(A\times B)\cup(C\times D)=\emptyset$$

but

$$(A\cup C)\times(B\cup D)=\{(1,1)\}\neq\emptyset$$

Problem 4. If f and g are *different* functions from a set A to a set B, show that $f \cup g$ is not a function.

Suppose f and g are different functions. Then there exists $a\in A$ s.t. $f(a)\neq g(a)$. Consider the relation $R=f\cup g$. Note that $(a,f(a))\in R$ and $(a,g(a))\in R$. Since there are two possible images for $a\in A$, R is not a function.

Problem 5. Let $A = \{0, 1\}$. Answer, with proof, the following two questions. On the set $\{0, 1\}$ with two elements, is there

(a) a relation R that is reflexive and symmetric, but not transitive?

No! If there were such a relation R, then R would include $\{(0,0),(1,1)\}$ since it is reflexive. To make it not-transitive we should add either $(0,1)$ or $(1,0)$. On the other hand, to keep R symmetric we have to add both $(0,1)$ and $(1,0)$. Then R has to be $A \times A$ which is transitive. Hence there is no way we can find such a relation.

(b) a relation T that is symmetric and transitive, but not reflexive?

Yes! $T = \{(0,0)\}$. T is not reflexive because $(1,1) \notin T$. It is clear that T is symetric and transitive.

Problem 6. Assume $f\colon A \to B$ and $g\colon B \to C$. If $g \circ f$ is one-to-one (that is, injective), must g be one-to-one? Prove your answer.

No! Consider the following counter-example

$A = \{1\}$, $B = \{1,2\}$, $C = \{1\}$, $f = \{(1,1)\}$, $g = \{(1,1),(2,1)\}$

The composition $g \circ f = \{(1,1)\}$ is 1-1 but g is not one-to-one.

Problem 7. Suppose $f\colon A \to B$, and f has the property that, for all subsets X of A,

$$f[X^c] = (f[X])^c.$$

(Here $f[X] = \{f(y) : y \in X\}$, also denoted by $f(X)$.)

(a) Show that f is surjective (that is $f[A] = B$). (*Hint:* Consider $X = \varnothing$.)

$$f(A) = f(\phi^c) = f(\phi)^c = \phi^c = B$$

(b) Show that f is injective. (*Hint:* If $d \neq e$ in A, show $f(d) \notin f[\{e\}]$.)

(To show f is injective, we should show

$f(d) = f(e) \Rightarrow d = e$

Alternatively we can use

$d \neq e \Rightarrow f(d) \neq f(e)$)

Pick $d, e \in A$ s.t $d \neq e$. Then $d \notin \{e\}$ and $d \in \{e\}^c$
It follows that $f(d) \in f(\{e\}^c) = f(\{e\})^c$. Thus
$f(d) \notin f(\{e\})$. As a result of this, we have $f(d) \neq f(e)$
Therefore f is injective.