Name:

Student number:

2010

# METU MATH 111, Final Exam

Monday, January 10, 2011, at 16:30 (120 minutes)

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**Instructions:** There are 8 numbered problems on 4 pages. It should be obvious to the grader how to read your solutions. Please work carefully.

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**Problem 1.** True or false? (Put $T$ or $F$ in the brackets. Do *not* justify your answers. Blank answers get 0; *wrong* answers get negative points.)

(a) $\mathbb{R}$ is countable ........ (F)

(b) The set of polynomials in the variable $x$ with integer coefficients is countable ........ (T)

(c) If $A$ and $B$ are uncountable sets, then $A \smallsetminus B$ is countable ........ (F)

(d) A relation $R$ is symmetric if and only if $R = \breve{R}$ (that is, $R = R^{-1}$) ........ (T)

(e) For all sets $A$ and $B$, $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$ ........ (T)

(f) For all sets $A$ and $B$, $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$ ........ (F)

(g) $\neg(\exists x \in A)\ P(x)$ is logically equivalent to $(\forall x \in A)\ \neg P(x)$ ........ (T)

(h) $((\forall x \in A)\ P(x)) \wedge ((\forall x \in B)\ P(x))$ is equivalent to $(\forall x \in (A \cup B))\ P(x)$ ........ (T)

(i) If $A$ is a *totally* (or linearly) ordered set and $x$ is a maximal element of $A$, then $x$ is the maximum element of $A$ ........ (T)

**Problem 2.** The triangle inequality in $\mathbb{R}$ is given by $\forall x\, \forall y\ (|x+y| \leqslant |x| + |y|)$. Using this, prove that

$$|a_0 + a_1 + \cdots + a_n| \leqslant |a_0| + |a_1| + \cdots + |a_n|$$

for all nonempty finite sets $\{a_0, a_1, \ldots, a_n\}$ of real numbers.

Let $P(n)$ be the statement

$$|a_0 + \cdots + a_n| \leq |a_0| + \cdots + |a_n|$$

Since $|a_0| \leq |a_0|$ for any $a_0 \in \mathbb{R}$, $P(0)$ is true. Now suppose that $P(n)$ is true. Then

$$|\underbrace{a_0 + \cdots + a_n}_{x} + \underbrace{a_{n+1}}_{y}| \leq |a_0 + \cdots + a_n| + |a_{n+1}| \quad \text{(Triangle Inequality)}$$

$$\leq |a_0| + \cdots + |a_n| + |a_{n+1}| \quad \text{(Inductive Hypothesis)}$$

Thus $P(n+1)$ is true. By induction we conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

**Problem 3.** Suppose $R$ is a partial ordering of $A$, and $S$ is a partial ordering of $B$. Let

$$T = \{((a, b), (a', b')) \in (A \times B) \times (A \times B) \colon a\, R\, a' \text{ and } b\, S\, b'\}.$$

(a) Show that $T$ is a partial ordering of $A \times B$.

T is reflexive: Since R and S are partial orders $aRa$ and $bSb$ for $a,b$ in $A, B$ respectively. Thus $(a,b)T(a,b)$.

T is antisymmetric: If $(a,b)T(a',b')$ and $(a',b')T(a,b)$ then $aRa'$ and $a'Ra$. Thus $a=a'$ since R is antisym. Similarly $b=b'$. Therefore T is antisymmetric.

T is transitive: If $(a,b)T(a',b')$ and $(a',b')T(a'',b'')$, then $aRa'$ and $a'Ra''$. Thus $aRa''$ since R is transitive. Similarly $bSb''$. Therefore $(a,b)T(a'',b'')$ and T is transitive.

(b) If both $R$ and $S$ are *total* (or linear) orderings, will $T$ also be a total ordering? Justify your answer.

No. Consider $A=B=\mathbb{R}$ and "less than or equal" relation on both A and B. The induced relation T is not a total order because neither $(1,2)T(2,1)$ nor $(2,1)T(1,2)$.

**Problem 4.** Suppose $b \in \mathbb{N}$, and $A$ is a nonempty subset of $\mathbb{N}$ such that, for every element $x$ of $A$, we have $x \leqslant b$. Prove that $A$ has a maximum element with respect to $\leqslant$.

Consider $C=\{b-x : x\in A\}$, a subset of $\mathbb{N}$ since $x\leq b$ for all $x\in A$. Using well ordering principle pick $b-a_0 \in C$ a smallest element of C. Then $b-a_0 \leq b-x$ for all $x\in A$. It follows that $x\leq a_0$ for all $x\in A$. The element $a_0$ is a maximum element of A with respect to $\leq$.

**Problem 5.** Define the relation $E$ on $\mathbb{R}$ by $x \, E \, y \iff x - y \in \mathbb{Z}$.

(a) Show that $E$ is an equivalence relation.

E is reflexive: For any real $x$, we have $x - x = 0 \in \mathbb{Z}$. Thus $x E x$. The relation E is reflexive

E is symmetric: If $xEy$, then $x-y=k$ for some $k \in \mathbb{Z}$. Then $y-x=-k \in \mathbb{Z}$. So $yEx$ and E is symmetric

E is transitive: If $xEy$ and $yEz$, then we have $x-y=k$ and $y-z=l$ for some $k, l \in \mathbb{Z}$. Now $x-z$ equals $k+l$, an element of $\mathbb{Z}$. Therefore $xEz$ and E is transitive

(b) What is [0]? $[0] = \{x \in \mathbb{R} : x - 0 \in \mathbb{Z}\} = \mathbb{Z}$

(c) Write three distinct elements of $\mathbb{R}/E$. $[0], [1/3], [2/3]$

(d) Is $\mathbb{R}/E$ countable? (Explain briefly.)

For any $x \in [0,1)$, we obtain a distinct equivalence class of E. Since $[0,1)$ is uncountable, $\mathbb{R}/E$ is uncountable too.

**Problem 6.** Let $S$ be the relation $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$ on $\mathbb{R}$. Answer each of the following questions by giving either a proof or a counterexample.

(a) Is the logical sentence $\forall x \, \exists y \, (x \, S \, y)$ true in $\mathbb{R}$?

No. If $x=2$, then there is no $y \in \mathbb{R}$ s.t $x^2+y^2=1$.

(b) Is $S$ a function from $[-1, 1]$ to $\mathbb{R}$?

No. If $x=0$, then consider $y_1=1$ and $y_2=-1$. We have $xSy_1$ and $xSy_2$ but $y_1 \neq y_2$.

**Problem 7.** Explain briefly whether there are there propositional formulas $F$ and $G$ such that:

$$(P_0 \,\&\, P_1) \vee F \sim P_0 \vee P_1, \qquad (P_0 \vee P_1) \vee G \sim P_0 \,\&\, P_1$$

If $F$ is the formula $P_0 \vee P_1$ then the former equivalence holds.

There is no propositional formula $G$ satisfying the latter equivalence. To see this consider $P_0 = 0$ ad $P_1 = 1$. Then the left hand side is always $1$ whereas the right hand side is always $0$.

**Problem 8.** Do the the following equations (where $a$ and $b$ range over $\mathbb{Z}^+$) define functions $f$ and $g$ from $\mathbb{Q}^+$ to $\mathbb{Q}^+$? Justify your answers.

$$f\left(\frac{a}{b}\right) = a + b, \qquad g\left(\frac{a}{b}\right) = \frac{2a^2 + ab + b^2}{2b^2}.$$

The former equation does not define a function because $\frac{1}{2} = \frac{2}{4}$ in $\mathbb{Q}^+$ but

$$f\left(\frac{1}{2}\right) = 1+2 \neq 2+4 = f\left(\frac{2}{4}\right).$$

We can rewrite the latter equation as follows:

$$g\left(\frac{a}{b}\right) = \left(\frac{a}{b}\right)^2 + \frac{1}{2}\left(\frac{a}{b}\right) + \frac{1}{2}$$

Changing $\frac{a}{b}$ with $\frac{c}{d}$ (where $ad = bc$) has no effect on the value of the right hand side. Hence $g$ defines a function from $\mathbb{Q}^+$ to $\mathbb{Q}^+$. Indeed we have $g(x) = x^2 + \frac{1}{2}x + \frac{1}{2}$ for all $x \in \mathbb{Q}^+$