Middle East Technical University - Department of Mathematics
Math 473 - Ideals, Varieties and Algorithms
Midterm 2 - December 26, 2017

Name:________________________ ID Number:____________

1. ($4\times 5pts.+10pts.$) The tangent space to the twisted cubic in $\mathbb{C}^3$ is parametrized by $x = t + u$, $y = t^2 + 2tu$ and $z = t^3 + 3t^2u$ where $(t, u) \in \mathbb{C}^2$ (t and u are the parameters). Using the lex order with $t > u > x > y > z$, the reduced Groebner basis of the ideal $I = \langle x - t - u,\ y - t^2 - 2tu,\ z - t^3 - 3t^2u\rangle \subset \mathbb{C}[t, u, x, y, z]$ consists of the following polynomials:
$g_1 = t+u-x$, $g_2 = u^2 - x^2 + y$, $g_3 = ux^2 - uy - x^3 + (3/2)xy - (1/2)z$,
$g_4 = uxy - uz - x^2y - xz + 2y^2$, $g_5 = uxz - uy^2 + x^2z - (1/2)xy^2 - (1/2)yz$,
$g_6 = uy^3 - uz^2 - 2x^2yz + (1/2)xy^3 - xz^2 + (5/2)y^2z$,
$g_7 = x^3z - (3/4)x^2y^2 - (3/2)xyz + y^3 + (1/4)z^2$.
a) Write down the generators of the elimination ideals I_1, I_2, I_3 and I_4.
b) What is the smallest variety in $\mathbb{C}^3$ which contains the image of the given parametrization above?
c) Which partial solutions in $V(I_2)$ extend to partial solutions in $V(I_1)$, and which partial solutions in $V(I_1)$ extend to solutions in $V(I)$?
d) Is $G = \{g_1, g_2, .., g_7\} \subset \mathbb{R}[t, u, x, y, z]$ a Groebner basis of $\tilde{I} = \{x - t - u,\ y - t^2 - 2tu,\ z - t^3 - 3t^2u\rangle \subset \mathbb{R}[t, u, x, y, z]$?
e) Show that any partial solution $(x, y, z) \in V(\tilde{I}_2) \subset \mathbb{R}^3$ extends to a unique solution $(t, u, x, y, z) \in V(\tilde{I}) \subset \mathbb{R}^5$.

2. ($20pts.$) Let a parametrization in k^3 be given by the equations $x = \dfrac{1 + t^2 - u^2}{t^3 + u^3}$, $y = \dfrac{2 - t^2 + 3u^2}{t^2u + tu^2}$ and $z = t^2 + ut + u^2$ where u and t are parameters. If we want to find the smallest variety in k^3 which contains the image of this parametrization, what is the procedure to follow. Write down the steps of the process clearly identifying which ideal you consider and what operations are performed. Number the steps as Step 1, Step 2 etc., and describe what should be done in each step without going through the calculations.

3. ($10 + 6pts.$) a) For any non-constant $f \in \mathbb{R}[x]$, prove that f has a repeated irreducible factor (there exists an irreducible polynomial $g \in \mathbb{R}[x]$ such that $g^2|f$ in $\mathbb{R}[x]$) if and only if the resultant $Res(f, f', x) = 0$ where f' is the derivative of f.
b) Show briefly that if we replace the field $\mathbb{R}$ in part (a) by $\mathbb{C}$, then we can replace the expression "repeated irreducible factor" by "repeated root".

4. ($8 + 8pts.$) Let $f = xy^3 + xy^2 + y^3 - xy + 2y^2 - 1$ and $g = -x^2y^3 + y^4 + 2x^2y + y^3 - x^2 + y - 1$ be two polynomials in $k[x, y]$ for some field k.
a) Write down $Res(f, g, x)$ as a determinant.

b) If $Res(f, g, x) \neq 0$, does that imply that f and g are relatively prime in $k[x, y]$? If not, what else should be checked in order to determine if they are relatively prime or not?

5. a) Let $f \in \mathbb{C}[x, y]$ be non-constant. Show that $V(f)$ is infinite.
b) For two non-constant polynomials $f, g \in \mathbb{C}[x, y]$ show that $V(f, g)$ is infinite if and only if f and g have a common irreducible factor in $\mathbb{C}[x, y]$. (Hint: Use resultants and their properties and also part (a).)