M E T U
Department of Mathematics

Field Extensions and Galois Theory Midterm II	
Code : *Math 368* Acad. Year : *2017-2018* Semester : *Spring* Instructor : *Karayayla* Date : *02.05.2018* Time : *17.40* Duration : *150 minutes*	Last Name : Name : Student No : Department : Signature : **7** Questions on **5** Pages SHOW DETAILED WORK!

1	2	3	4	5	6	7	8		

1.(12 pts.) Write down the definitions of a splitting field, a normal field extension, and a separable element of an algebraic field extension.

2.(12 pts.) Let $\alpha \in L$ be a root of $x^7 + \sqrt[5]{3}x^4 + ix^3 + \sqrt[3]{5} + 2$ where L is a field extension of $\mathbb{Q}$. Show that $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq 210$.

3.(2 × 7 pts.) a) Let $\beta = \sqrt{2+\sqrt{2}} \in \mathbb{R}$. Find the minimal polynomial f of β over Q .

b) Show that $L = \mathbb{Q}(\beta)$ is a splitting field of f over $\mathbb{Q}$.

4. (14 pts.) Show that $\mathbb{Q}(\sqrt{3}, \sqrt[3]{5})$ is not a splitting field of any $f \in \mathbb{Q}[x]$ over $\mathbb{Q}$.

5.(6 + 10 pts.) a) Write down a basis of $L = \mathbb{Q}(\sqrt{2}, i)$ over $\mathbb{Q}$ considering it as a vector space.

b) What is the condition on $k \in Q$ such that $\alpha = k\sqrt{2} + i$ is a primitive element of the extension $\mathbb{Q} \subset L$ (that is, $L = \mathbb{Q}(\alpha)$). (Hint: Use the basis from part a and use linear algebra.)

6.(4 + 6 + 6 pts.) Let F be a field of characteristic $p > 0$ and assume that $f = x^p - x + c \in F[x]$ is irreducible over F.

a) Show that f is separable over F.

b) Show that if α is a root of f in some extension field L over F, then $\alpha + 1$ is also a root of f.

c) Show that $F(\alpha)$ is a normal extension field over F.

7.(2×8 pts.) Let $L = \mathbb{Q}(\sqrt[5]{3}, \zeta_5)$ where $\zeta_5 = e^{\frac{2\pi i}{5}} \in \mathbb{C}$.

a) For a $\sigma \in Gal(L/\mathbb{Q})$, list the possible values of $\sigma(\sqrt[5]{3})$ and $\sigma(\zeta_5)$.

b) How many elements does the Galois group $Gal(L/\mathbb{Q})$ have?

8. (Bonus: 10 pts.) For a field F of characteristic $p > 0$, the Frobeneous map $\varphi : F \to F$ is defined by $\varphi(x) = x^p$, and it is a field homomorphism.
Assuming that $\varphi : F \to F$ is onto, prove that any irreducible polynomial $f \in F[x]$ is separable over F.