Middle East Technical University - Department of Mathematics
Math 473 - Ideals, Varieties and Algorithms
Midterm 1 - November 22, 2017

Name:________________________ ID Number:____________

1.(10 pts.) Let $g = 1 + y + 2xyz + 3x^3y + xz + y^2z + y^2z^2$, $f_1 = z + y^2$, $f_2 = yz + 1$ and $f_3 = xz + y$ be polynomials in $k[x, y, z]$ for a field k. Using the *grlex* order with $x > y > z$, perform the Division Algorithm to divide g by the triple (f_1, f_2, f_3) (in this order).

2.(3x6 pts.) For a monomial ordering $>$ on monomials of $k[x_1, x_2, ..., x_n]$ prove each of the following statements. (Notation: $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ where $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n_{\geq 0}$.)
a) $x^\alpha > x^0$ for all $\alpha \in \mathbb{Z}^n_{\geq 0}$. (0 in x^0 stands for $(0, 0, ..., 0)$.)
b) $x^\alpha | x^\beta$ implies $x^\alpha < x^\beta$.
c) If $I = \langle x^\alpha | \alpha \in A\rangle$ is a monomial ideal for some $A \subset \mathbb{Z}^n_{\geq 0}$ such that $A \neq \emptyset$, and if x^δ is the minimum element of the set S of all monomials of I, then show that $\delta \in A$.

3.(16 pts.) Let $W \subset k^n$ be an affine variety. If $I(W)$ denotes the ideal of the variety W and $V(J)$ denotes the variety of an ideal $J \subset k[x_1, x_2, ..., x_n]$, then show that $V(I(W)) = W$.

4.(16 pts.) Let G and $\tilde{G}$ be Groebner bases for the same ideal $I \subset k[x_1, x_2, ..., x_n]$ for a fixed monomial ordering. Show that $\overline{f}^G = \overline{f}^{\tilde{G}}$ for all $f \in k[x_1, x_2, ..., x_n]$.

5.(10+5 pts.) Let $S = \{f_i \in k[x_1, x_2, ..., x_n] \,|\, i \in \mathbb{Z}^+\}$ be a countably infinite collection of distinct polynomials.
a) If $I = \langle f_1, f_2, ..., f_i, ...\rangle$ is the ideal generated by S in $k[x_1, x_2, ..., x_n]$, then show that there is $N \in \mathbb{Z}^+$ such that $I = \langle f_1, f_2, ..., f_N\rangle$.
b) Is there an algorithm to find such a number N whose existence is shown in part (a)? Explain.

6.(20+5 pts.) a) Apply Buchberger's Algorithm to find a Groebner basis for $I = \langle x^2y - 1, xy^2 - x\rangle$ using *lex* order with $x > y > z$.
b) Find the reduced Groebner basis of I.